High Rate QC-LDPC Codes with Optimization of Maximum Average Girth

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Abstract —In this paper, the construction of the high rate Quasi-Cyclic Low-Density Parity-Check (QC-LDPC) codes is presented for the wired or wireless optical communications with high frequency spectrum efficiency, especially at rather good channel environments. The good code performance can be achieved by an improved girth search algorithm, where the girth is optimized by the maximization of the average loop length. It mainly searches the optimal circulant shift offsets in each permutation sub-matrix to maximize the average girth of the variable nodes in the Tanner graph. And the property of the maximum average girth can be exploited in the classic Belief Propagation (BP) LDPC decoding to avoid the unnecessary false message self-feedback by the short loops in the iterative decoding as far as possible. Meanwhile, the proposed codes are also optimally designed for good girth and degree profile with conventional Progressive Edge Growth (PEG) and Extrinsic Information Transfer (EXIT) chart techniques. Simulation results show that the proposed girth optimized QC-LDPC codes obtain a little better Bit-Error-Rate (BER) performance, when compared with the current channel codes of similar code parameters used in the optical communications. Therefore, the proposed code design methodology can be effectively adopted in the construction of the high rate QC-LDPC codes, which can be used in the optical communications for good coding gains.

Index Terms—QC-LDPC codes, girth optimization, average girth

I. INTRODUCTION

Nowadays, modern digital communication systems develop rapidly with larger capacity and high data rate to fulfill the requirement of mass multimedia information transmission and so on. One of such systems, the optical communication system, has several advantages, such as ultra larger capacity and better transmission channel, when compared with wireless and even the traditional cable communications. And it has been widely used for the long-haul transmission all over the world. But the reliability and effectiveness are still crucial for the optical communications. And effective measures are needed to guarantee the good efficiency and reliability. One of such measures is the Error Control Code (ECC), which plays a key role on the ever increasing demand of high data rate transmissions and good quality of reliable service at low cost. Previously, the traditional ECC codes used in the optical communications are mainly the Hamming codes, the Bose-Chaudhuri-Hocquenghem (BCH) codes and the Reed-Solomon (RS) codes [1]. They have been adopted in the ITU-T standard and widely used in some other communication applications. But they are less efficient channel codes in practice. Due to the milestone in coding theory as the discovery of modern channel codes, *e.g.* the Turbo codes and the Low-Density Parity-Check (LDPC) codes, they have come into the new era of more powerful Shannon’s capacity with iterative decoding of moderate complexity. LDPC codes are one class of such capacity approaching ECCs, which were firstly discovered by Gallager in 1962 [2]. Because of the limitation of electronic technologies at that time, they weren’t widely investigated and developed until the middle 1990s, when they were rediscovered by MacKay, *et al* [3]. Accompanied by the tremendous improvements in the computing capability by the latest electronic technologies, LDPC codes have been designed, constructed and even implemented in the communication practice. Also more attentions have been focused on the high rate LDPC codes for high spectrum efficiency and low coding complexity in the optical communications under good transmission circumstances. The parity check matrices of the LDPC codes can be represented by a bipartite graph, also called Tanner graph [4], [5], which greatly eased the analyses of them. Moreover, good performance of different classes of the codes has been assessed in practice [1]. The results show that they were extremely effective with high performance and acceptable complexity in the optical communication systems. For example, the achieved coding gains of them are much larger than those of traditionally widely used RS codes, convolutional codes, even the Turbo codes, and so on [6]. So the design of the LDPC codes for using in the optical communications has been the hot-spot recently.

LDPC codes are powerful channel codes. But their encoding complexity is large and in proportion to the length of the code, which prohibits the use of codes with long lengths. So it is necessary to construct a new kind of structured codes with both low complexity and good

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performance. The Quasi-Cyclic LDPC (QC-LDPC) codes belong to the structured codes, the matrix structures of which are much more regular and simpler than those original LDPC codes with random check matrices. The sub-matrices in the parity-check matrix **H** of the QC-LDPC codes mainly consist of the Circulant Permutation Matrix (CPM) or zero square matrix. And the CPM can be easily implemented by the efficient shift registers in the hardware implementation, since it has fixed column and row weight and each row in the matrix is the cyclic shift of the first row. Also due to the sparse property of **H**, linear complexity of both encoding and decoding are easily obtained. In practice, only the positions and the cyclic shift numbers of the non-zero CPM are required to be stored in the onboard memory, which greatly reduces the storage resources. Recently, the studies in the construction of the QC-LDPC codes of high rate has also become the hot spot in the channel coding application of the optical communication systems [1]. Representative methods of the code construction are based on finite geometries and finite fields in [7]-[10]. It mainly introduced a CPM design in the finite fields to obtain good code properties of both good threshold and low error-floor. But the analyses of them are rather complex and hard to be directly perceived through the sense. Also the large column weight can be used in code design to get the deterministic girth-eight QC-LDPC codes for good performance [11]-[13]. But it increases the 1's density in the sparse check matrix and thus raises the decoding complexity. In [8], the algebraic QC-LDPC codes were generated for low error-floor, large girth, as well as a reduced-complexity decoding scheme. However, due to the fixed algebraic structures in designing the codes, the code parameters can not be easily adjusted for flexible code rate and code length. Since pragmatic codes must have finite length, there are usually loops in their Tanner graphs under some optimal degree profiles. And the degree profiles are the portion of variable and check node degree to the total degree. For an LDPC code to achieve the performance of Maximum Likelihood (ML) decoding, there should not be any loop in the Tanner graph of the code. However, due to the existence of short loops in the code graph, the update of the variable node message will be confronted with the problems of the self-feedback message propagation of the false information in the iterative Belief Propagation (BP) decoding. Then the requirement of the statistical independence of all the message propagation can not be satisfied, which deteriorates the code performance. So the short girth, which increases the self-feedback message passing in decoding, should be avoided as far as possible. In this paper, the notation of the maximum average girth is defined as an objective function and it is used to optimize the girth property of all variable nodes in the code design. And a maximum average girth searching algorithm is utilized here to optimize the structures of the QC-LDPC codes for good loop property [14]. Also the well-known PEG [15] and the EXIT [16] techniques are used to obtain the optimal degree profiles for good decoding threshold and low error floor.

The remainder of the paper is organized as follows. In Section II, the influence of short cycles in a Tanner graph of the LDPC code is illustrated. In this section, the girth problem related to the loops is put forward and the effective loop optimization method in designing the codes, *e.g.* PEG algorithm, is introduced too. In order to meet the requirements of the optical communication systems, *i.e.*, high rate for high efficiency and low complexity for easy implementation, a girth optimized scheme for the construction of the high-rate QC-LDPC codes is proposed and analyzed in Section III. Section IV presents the simulation results of the code constructed by the proposed method, as well as the performance analyses. Finally, the summary is concluded in Section V.

II. The Influence of Loops in Tanner Graph

Tanner graphs belong to bipartite graphs with two classes of node sets and several edges between the nodes of them. The LDPC code is then easily represented by the graph. A Tanner graph can be expressed as a graph $G=\{(V, E)\}$, where V and E are the sets of nodes and edges respectively. It includes the non-intersect variable and check node sets as $V=V_b \cup V_c$, where $V_b=(v_0, v_1, ..., v_{n-1})$ is the variable node set and $V_c=(c_0, c_1, ..., c_{m-1})$ is the check node set. And they correspond to the column and row vector in the check matrix **H**, respectively. If the element in the i-th row and the j-th column of **H** is not 0, there is the expression $\mathbf{H}_{i,j}\neq 0$. It means that there are one or more than one edge between the j-th variable node and the i-th check node in the Tanner graph. In the classic Tanner graph, only one edge between the two different types of the nodes is allowed. The multi-edge bipartite graph with parallel edges can also be extended to the original Tanner graph by the "copy-and-permute" operation, which can also be analyzed here after extension. A path in G of length k is defined as the trace, which goes through the node sequence $\{v_1, c_2, ..., c_{k-1}, v_k\}$. If it ends at the same node, *i.e.*, $v_1=v_k$, the trace must form a loop in the Tanner graph. The number of the edges in a loop is called loop length. And the shortest loop length in a Tanner graph is defined as the girth of the code [17], which usually decides the code performance to some extent. And the girth should be optimized to reduce the performance degradation caused by false message self-feedback propagation in the short cycles. The girth value and the short loops are key factors affecting the code performance. And it can be extended in any manner of tree to illustrate the possible loop. One typical Tanner sub-graph of the variable nodes extending to depth-l is shown in Fig. 1.

In the tree extension of the Tanner sub-graph, one variable node is usually chosen as the root node. And the depth is counted with every pair of variable and check nodes together. From Fig. 1, the minimum number of the edges between two nodes is called the distance between them, and the distance between the root node and the nodes first appearing in the depth-k is $2k$ (from the root node to the variable nodes in the layer of depth-k) or $2k+1$ (from the root node to the check nodes in the layer of depth-k). If there are two check nodes in depth-k connected with the same variable node in depth-k, there must be at least a loop. And the corresponding loop length is $4k$. And if two variable nodes in depth-k connected with the same check node in depth-k+1, then the corresponding loop length is $4k+2$.

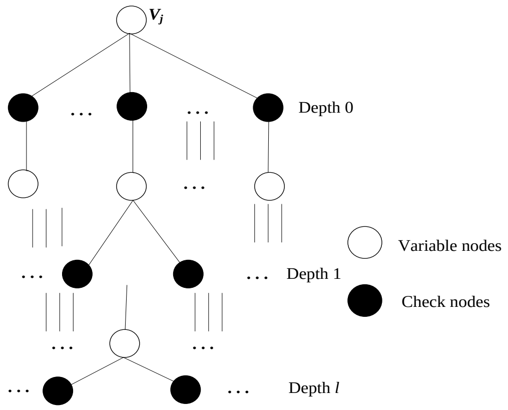

Fig. 1. A Tanner sub-graph extended with a root (variable) node v_j.

The BP decoding can achieve the optimal maximum likelihood decoding solution of an LDPC code with a cycle-free Tanner graph of the code. But in practice, the cycle-free requirement is usually hard to be achieved, due to the limited (short or moderate) code length constrained by the coding complexity. At the same time, the short cycles will cause much more performance deterioration than larger loops, where the false message in a variable node will pass much more edges to self-feedback itself. So the codes are required to be constructed with largest possible girth under finite code length in coding practice. Followed by this criterion, more performance gain can be obtained when there are less loops or more average loop length in the Tanner graph of the code. It can also be explained that more reliable and independent extrinsic information is received and exchanged in the iterative decoding. And the heuristic PEG algorithm can be used to construct the code by adding every edge one by one, to guarantee the largest possible loop length in the Tanner graph of the code [15]. Then the code performance can be improved by more accurate decision in BP decoding. In general, larger girth in the design of a QC-LDPC code will lead to better code performance.

III. Girth Optimization for QC-LDPC Codes

The general form of QC-LDPC codes is also known as the Block-LDPC codes, the parity-check matrix **H** of which is expressed as (1). In (1), $I(p_{i,j})$ is the CPM with block index i=0, 1, 2, ..., m-1 and j=0, 1, 2, ..., n-1. The size of the CPM is $L\times L$ and $p_{i,j}$ is the circulant shift factor, which is mainly generated by the cyclic-right shifting of a standard identity matrix. If the matrix element is less than 0, *i.e.* $p_{i,j}<0$, $I(p_{i,j})$ represents an all-zero square matrix. Otherwise, $I(p_{i,j})$ is formed by a cyclically shifting the identity matrix to the right $p_{i,j}$ position with the integer $p_{i,j}$, where $0\le p_{i,j}\le L-1$. According to the definition, the CPM with cyclic shift factor 0 and L is same, and the CPM with cyclic shift factor 1 is same as CPM with L+1 and so on.

$$\mathbf{H}=\begin{bmatrix} I(p_{0,0}) & I(p_{0,1}) & \cdots & I(p_{0,n-1}) \\ I(p_{1,0}) & I(p_{1,1}) & \cdots & I(p_{0,0}) \\ \vdots & \vdots & \ddots & \vdots \\ I(p_{m-1,0}) & I(p_{m-1,1}) & \cdots & I(p_{m-1,n-1}) \end{bmatrix} \tag{1}$$

The QC-LDPC code, *i.e.* Block-LDPC code, can be simplified to a base matrix **B**. The elements in matrix **B** are the cyclic shifting factor of each CPM in the parity-check matrix **H**. And it is given in (2).

$$\mathbf{B}=\begin{bmatrix} p_{0,0} & p_{0,1} & \cdots & p_{0,n-1} \\ p_{1,0} & p_{1,1} & \cdots & p_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m-1,0} & p_{m-1,1} & \cdots & p_{m-1,n-1} \end{bmatrix} \tag{2}$$

For matrix **B**, when $p_{i,j}\ge 0$, it is replaced by '1'. Otherwise, it substitutes for '0'. And the corresponding matrix is defined as **B**(**H**). For the base matrix **B**(**H**), the number of all possible combinations of the shift factor set is L^z , where z is the number of element '1' in the base matrix and L is also known as the extension factor. Then the computational complexity of the optimal solution grows exponentially with z, and it is hard to be obtained. So a sub-optimal scheme is proposed here to solve the matrix extension problem.

According to the extended definition of girth in [17], each node has the girth, and it is represented as the node girth g [14], [18]. Based on this notation, given the column length of the parity-check matrix **H** as N, the average girth $\bar{g}$ is defined as the girth averaged over all variable nodes. It is expressed as

$$\bar{g}=\sum_{j=0}^{N-1}\frac{g_{v_j}}{n_{v_j}} \tag{3}$$

where g_{v_j} is the girth of variable node v_j, and n_{v_j} is the number of variable node with shortest variable node girth g_{v_j} . Then, the average girth of the QC-LDPC codes $\bar{g}$ can be represented as

$$\bar{g}=\sum_{j=0}^{n-1}\frac{L\cdot g_{v_{j\times L}}}{n_{v_{j\times L}}} \tag{4}$$

where n is the column size of the base matrix **B**. Since the variable nodes in a CPM of the QC-LDPC code have the same girth distribution, the equation in (4) is established.

Based on the above girth analyses, the construction of a QC-LDPC code can be listed as follows. The aim of the algorithm is to achieve the maximum average girth in the corresponding Tanner graph by optimizing the set of the circulant shifting values of the CPMs in the base matrix. In the process of searching the optimal circulant shifting sets in each CPM to maximize the average girth, the matrix **B(H)** is initially sorted at first. The weight of each column is the number of '1' in this column. And the column vector is prepared in the non-decreasing order, *i.e.*, if $i>j$, the weight v_j is not greater than v_i. Due to (4), the average girth calculation is then simplified to calculate the girth of the last column in the CPM only. After that, it searches the optimal cyclic shifting factor in the sequence from left to right, top to bottom and one by one. Then the sub-optimal solution is obtained. If the cyclic shifting factor of a CPM is determined, the value will not affect by others. In other words, there is no repeated calculation in the whole calculation process. Next the complexity of the optimal and sub-optimal algorithm is compared.

For the optimal solution, L times search calculation are needed for each element "1" in **B(H)**, and the complexity of each search calculation is correlated with the local girth. The local girth is assumed as g, the corresponding iteration is $g/2$ in each search calculation. Based on the proposed algorithm, for a node with degree d, the addition and comparison operations are d-1 and d respectively. And the matrix **H** is filled block-by-block, the degree of each node is increasing, so the computational complexity of each search calculation grows with degree.

For the sub-optimal solution, the maximal node degree is adopted, and the maximal local girth of each node is calculated. Then the addition and comparison operation numbers in each search calculation are

$$g_{\max}(d_c-1)M/2+g_{\max}(d_v-1)N/2 \tag{5}$$

$$g_{\max}d_cM/2+g_{\max}d_vN/2 \tag{6}$$

where $g_{\max}$ is the maximal local girth of each node, d_v and d_c are the degrees of variable and check node respectively. The numbers of element 1 in **H** is defined as Z, then $Z=d_cM=d_vN$ is obtained. From (5), (6) and $Z=d_cM=d_vN$, we know that the search complexity of each shift factor is linear with 1's number in **H**, then the complexity is denoted as $O(Z)=aZ$. In order to achieve all shift factor, $z\times L$ times search calculation are needed, so the overall complexity can be expressed as $z\times L\times aZ$, where $Z=z\times L$, then the computation complexity of the sub-optimal solution can be reduced to $O(z^2)$. For general searching shift factors algorithms, just as the Cycle Elimination (CE) algorithm proposed by Lei Yang [19], the cycles, which are smaller than the objective girth, are eliminated. But when the objective girth is too large, the algorithm is not applicable. Because all non-zero elements in the base matrix involving in the cycles are checked and recorded, the computational complexity will be higher than that of the proposed sub-optimal solution.

Based on the above analyses, the detailed procedures of the maximum average girth optimization for a QC-LDPC code matrix can be summarized as below.

Step 1.1 The size of the base matrix **B(H)** is $m\times n$, and the extension factor is L. The parity-check matrix **H** is initialized as an all-zero matrix, whose dimension is extended to $mL\times nL$.

Step 1.2 For the base matrix **B(H)**, based on the column weights, the column vector is re-arranged into a non-decreasing order.

Step 1.3 To reorder the matrix data, assume the row and column index of the j-th element '1' in i-th column are r_b, c_b, respectively, where i = 1, 2, ..., n, j = 1, 2, ..., m. With the extension factor L, there are L kinds of CPM in the whole matrix **B(H)**. For $0\leq k\leq L$-1, the CPM with circulant factor k is added to the range $(r_b$-1$)L$+1 to r_bL rows and $(c_b$-1$)L$+1 to c_bL columns in the matrix **H**. Using the message-passing algorithm to count the girth and the amount of the c_bL-th column in **H**, the c_bL-th column will be the last non-zero column in **H** [1]. Finally, the CPMs with the proper circulant shifting factors are added in the final check matrix **H**, which makes the solution of (4) to be maximized in the corresponding position of **H**. And the parity-check **H** is thus generated, when all the optimal circulant shifting factors are searched completely.

As described above, the global shortest loop is the key factor affecting the performance of the LDPC codes. So the proposed QC-LDPC codes should also be designed to avoid them as much as possible, especially the length-4 loops. Also the degree profiles of the LDPC codes greatly affect the decoding threshold [20]. And the EXIT chart technique [16] can be adopted here to obtain the optimized degree profiles efficiently. Then the complete construction of the high rate QC-LDPC codes can be described as follows:

Step 2.1 The extrinsic information transfer (EXIT) chart is used to determine the optimal check node distribution, the number of the variable node and the corresponding degree distribution.

Step 2.2 Based on the node degree distribution, it use Progressive Edge-Growth (PEG) algorithms produces the girth optimized base matrix **B(H)**.

Step 2.3 For the base matrix **B(H)**, it searches the shifting factors, which is set to maximize the average girth. Then the CPM of **B(H)** with the shifting offset set, which leads to the maximization of the average girth, is added to H, and the parity-check matrix is then obtained.

In the **Step 2.1**, the EXIT chart technique [16] is used to design the optimal degree profile for good decoding threshold. Generally, the design of the node degree profiles of the LDPC codes with EXIT chart is to evaluate the matching degree between the Variable Nodes Decoder (VND) curve and the Check Nodes Decoder (CND) curve. Shorter distance between the two curve leads to the low decoding threshold, and thus better threshold can be obtained for good decoding performance.

Due to the fairly good channel characteristics of the optical communication systems, the high rate LDPC codes are designed. Let m=4, n=24, the code rate is $R=(n-m)/n=5/6$. According the EXIT chart and the parameters described in [16], an optimized degree profile of an LDPC code with R=5/6 is given in Fig. 2, which shows the EXIT curve fitting. The optimal variable node degree distribution parameters are designed in Table I.

TABLE I: THE DEGREE DISTRIBUTION OF THE VARIABLE NODE

$d_{v,i}$	a_i	b_i
2	0.3836	0.2557
3	0.2329	0.2329
4	0.3835	0.5114

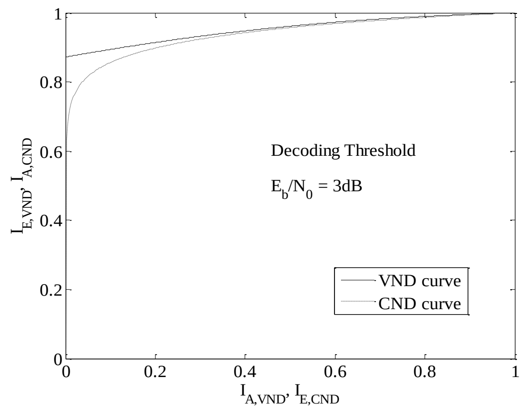

Fig. 2. EXIT curve for an LDPC code with code rate 5/6.

The Shannon capacity of a code with R=5/6 on an AWGN channel is 1.156 dB. By the density evolution (DE) method [19], the decoding threshold for the code of rate 5/6 is 2.626dB. In Fig. 2, the decoding threshold is got at about 3dB (E_b/N_0=3dB), when the VND curve matches the CND curve closely. So there are still 1.844 dB gap between the threshold by the EXIT method and the Shannon capacity. But the difference between the EXIT method and the DE method is rather trivial and the gap is only 0.374 dB.

After choosing check and variable degree d_c and d_v, the PEG algorithm can be used to generate the general framework of the base matrix. Since the constructed codes by the proposed algorithm are QC-LDPC codes, the encoding complexity of them is linear with the code length N. However, for the traditional LDPC codes randomly constructed by parity-check matrices, the encoding complexity is in proportion to N^2, *i.e.*, O(N^2). When compared with the random constructed LDPC codes, the proposed QC-LDPC codes have quasi-cyclic structure, where only the shift values are saved. And the binary multiplication operation of the code's matrix can be realized by means of cyclic shift due to the QC structure. Therefore, when the CPM size is L, the storage of QC-LDPC code will be reduced to $1/L^2$ of that of random LDPC code, which saves a lot of storage and is very suitable for practical application.

IV. SIMULATION RESULTS AND PERFORMANCE ANALYSES

For the base matrix analyzed in Section III, a series QC-LDPC codes with different code length are constructed. Let L=50, 100, 200, 300 and 400, then the QC-LDPC code with code lengths 1200, 2400, 4800, 7200, 9600 are constructed, respectively. In order to analyze the Bit-Error-Rate (BER) performance of each code constructed in this algorithm, the Binary Phase Shift Keying (BPSK) modulation is adopted. In simulations, the LDPC decoding at each E_b/N_0 will run until a specified number (*e.g.* 20) of error frames occur or a total of 0.8 million trials have been executed. And the maximum iterations are 50 for each LDPC frame decoding.

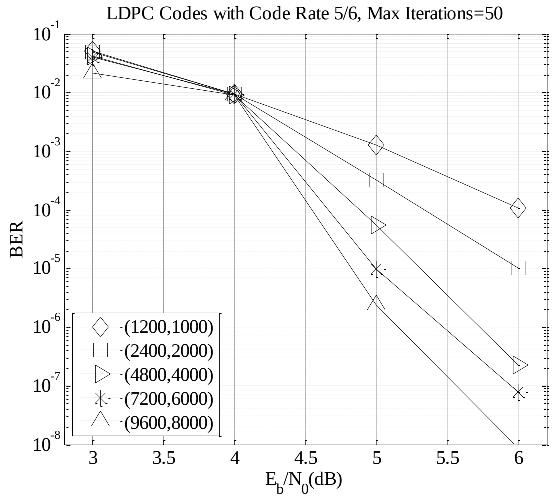

Fig. 3. The BER performance of the proposed LDPC code with same code rate 5/6 and different code length.

In Fig. 3, all these codes have same code rate 5/6, and the coding gains grows with the code length. At BER of 10^{-4}, the QC-LDPC code with code length 9600 has about 1.5dB, 1dB, 0.3dB and 0.1dB gains than the QC-LDPC code with length 1200, 2400, 4800 and 7200, respectively. Simulation results indicate that when the code length grows to a certain extent, the coding gains would not grow linear with it. According to [14] and [18], the LDPC codes with large code length always have large average girth, and thus the better BER performance. However, the increase of the code length brings much more computational complexity, which should be the additional cost for the improved performance. So a moderate code length is needed, which is compromised for the complexity of low encoding and decoding, and the good BER performance.

In a typical optical communication system, the QC-LDPC codes should have high coding gains, data rates, low error-floors and high iterative decoding convergence speeds. And the parity-check matrix of the code needs to be sparse for low complexity, where the number of '1' in the parity-check matrix must be far less than that of '0'. So each decoding iteration involves less calculation, and the decoding complexity will also decrease to some extent. To verify the effectiveness of the code design method, we simulate the whole system in an on-off keying (OOK) modulated optical communications with weak turbulence strength. So the channel can be approximately considered as the AWGN channel as used in [21].

The simulation parameters are set as follows. Let L=216, then a moderate length QC-LDPC code (5184, 4320) with code rate 5/6 can be constructed. According to the algorithm analyzed in section III, the QC-LDPC code with highest average girth is chosen. And the fraction of each loop length is achieved, as shown in Tab. II, where the minimum variable node girth of the code is 6, but it only accounts for a small fraction. With the same code length, code rate and degree distribution of the base matrix, the LDPC code guarantees full rank, LDPC code guarantees full rank and avoids cycle-4, are generated. They have average girth 9.67 and 10.29, respectively. Moreover, the comparable QC-LDPC codes (5184, 4320) constructed by the algorithm in [22] and [23] are given.

TABLE II: THE LOOP DISTRIBUTION OF THE QC-LDPC CODE (5184, 4320)

Loop length	Percentage (%)
6	2
8	9
10	38
12	51

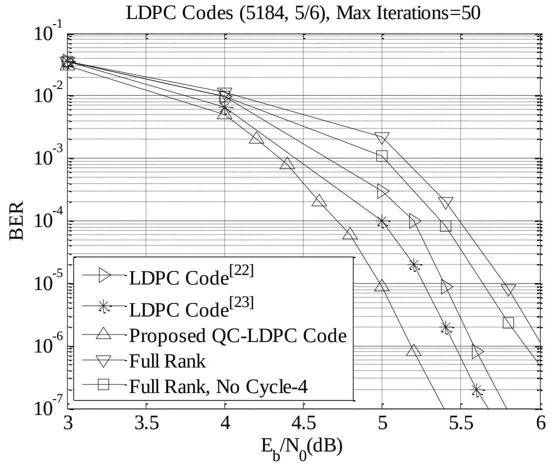

Fig. 4. The BER performance of the proposed LDPC code and the comparable codes with length 5184 and code rate 5/6.

Fig. 4 shows the performance of the proposed QC-LDPC code and other LDPC codes with the BP decoding algorithm. It indicates that the proposed code has a little better than the comparable codes. And the proposed code obtains about 0.7dB and 0.5dB performance gains than the codes with the same code parameters in [22], [23] at BER of 10^{-6}, respectively. Also, the proposed QC-LDPC code with highest average girth has better BER performance than the full rank LDPC code and LDPC code constructed with full rank and no cycle-4. At BER of 10^{-5}, it has about 1.3dB and 1.1dB performance gains, respectively. Therefore, it concluded that the girth conditioned codes outperform the unconditioned ones.

The proposed QC-LDPC codes are also compared with the codes used in optical communications [21]. In this case, the size of the CPM and the base matrix is convenient to be changed, since it can generate code of any rational code rate and length. According to the code construction described above, a QC-LDPC code with high code rate R=0.937 is produced as follows. Let the size of the base matrix be 4×63, and the extension factor L be 67. The QC-LDPC code (4221, 0.937) can be generated by the proposed code design method. Fig. 5 shows the BER of the proposed QC-LDPC code (4221, 0.937) and the QC-LDPC code (5334, 0.937) in [21] as well as the PEG-LDPC code (4221, 0.937) in [15] and MacKay-LDPC code in (4221, 0.937) [3] constructed by random construction method.

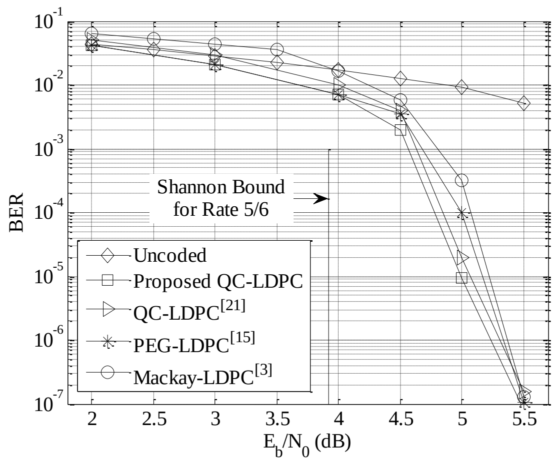

Fig. 5. The BER performance of the proposed LDPC code and the comparable codes with code rate 0.937 in the optical communications.

From Fig. 5, the coding gains of the proposed QC-LDPC code (4221, 0.937) is about 0.1 dB better than the QC-LDPC code (5334, 0.937) in [21] at BER of 10^{-6}. The QC-LDPC code in [21] is designed to avoid cycle-4 by choosing a prime extension factor. But the parity-check matrix **H** is a singular matrix. So the information node length is 3956 instead of 3953, and it also deteriorates some encoding complexity. Simultaneously, the PEG-LDPC code (4221, 0.937) [15] and MacKay-LDPC code (4221, 0.937) [3] have almost the same performance as the proposed QC-LDPC code (4221, 0.937), but the PEG-LDPC code (4221, 0.937) has error floor at BER of 10^{-7} [15]. And the MacKay-LDPC code (4221, 0.937) is constructed randomly, so the complexity of encoding and decoding is larger. For the proposed algorithm, it searches the optimal circulant shift sets of each CPM, as well as the optimal degree profile, to maximize the average girth and the decoding threshold, and thus it improves the performance better than that in [21]. The reason is that the smallest variable node girth, which increases the self-feedback message passing in decoding, only accounts for a small part in the proposed QC-LDPC codes. But the long loops of most variable node girths are 10 and 12, which will affect the performance much less. In addition, due to a sub-optimal solution applied here to approach the optimal solution, the computational complexity of the proposed QC-LDPC code construction is also less than those of the comparable codes.

V. CONCLUSION

This paper proposes an improved girth optimization algorithm for the construction of the high rate QC-LDPC codes. It mainly searches the optimal circulant shifting offsets in a circulant permutation sub-matrix to maximize the average girth of the code. In the algorithm, a sub-optimal solution is adopted for the low computational complexity and the flexibly adjustable parameters. So it has good properties of designing codes with both flexible code length and rate. Using the algorithm, a high rate 5/6 girth optimized QC-LDPC code with moderate code length 5184 is constructed, which possesses good performance, which is very suited for the applications in the optical communications. Simulation results show that the generated codes by the proposed algorithm show a little better performance gain than that of the contrast codes, and they have almost no error floor at BER of 10^{-7}. The coding gains of the proposed QC-LDPC code (4221, 0.937) is about 0.1 dB better than that of the QC-LDPC code in literature [21] with larger code length and full check matrix rank at BER of 10^{-7}. Therefore, the proposed algorithm is quite useful in the design of the efficiently high rate QC-LDPC codes, especially in the optical communications with high spectrum efficiency.

REFERENCES

[1] I. Djordjevic, W. Ryan, and B. Vasic, *Coding for Optical Channels*, New York, USA.: Springer-Verlag US, 2010, ch. 3, pp. 75-84.

[2] R. G. Gallager, *Low Density Parity Check Codes*, Cambridge, MA, USA.: MIT Press, 1963.

[3] D. J. MacKay, "Good error-correcting codes based on very sparse matrices," *IEEE Trans. on Information Theory*, vol. 45, no. 2, pp. 399-431, Mar. 1999.

[4] R. Tanner, "A recursive approach to low complexity codes," *IEEE Trans. on Information Theory*, vol. 27, no. 5, pp. 533–547, Sep. 1981.

[5] N. Wiberg, H. A. Loeliger, and R. Kotter, "Codes and iterative decoding on general graphs," *European Trans. on Telecommunications*, vol. 6, no. 5, pp. 513-525, Sept. 1995.

[6] W. E. Ryan and S. Lin, *Channel Codes: Classic and Modem*, 1st ed. New York, USA: Cambridge University Press, 2009, ch. 7, pp. 313-320.

[7] Y. Kou, S. Lin, and M. Fossorier, "Low-density parity-check codes based on finite geometries: A rediscovery and new results," *IEEE Trans. on Information Theory*, vol. 47, no. 11, pp. 2711–2736, Nov. 2001.

[8] J. Li, K. Liu, S. Lin, and K. Abde-Ghaffar, "Algebraic quasi-cyclic LDPC codes: Construction, low error-floor, large girth and a reduced-complexity decoding scheme," *IEEE Trans. on Communications*, vol. 62, no. 8, pp. 2626-2637, Aug. 2014.

[9] L. Zhang, S. Lin, K. Abdel-Ghaffar, *et al*., "Quasi-Cyclic LDPC codes on cyclic subgroups of finite fields," *IEEE Trans. on Communications*, vol. 59, no. 9, pp. 2330-2336, Sep. 2011.

[10] B. Zhou, J. Kang, S. Song, S. Lin, *et al*., "Construction of non-binary quasi-cyclic LDPC codes by arrays and array dispersions," *IEEE Trans. on Communications*, vol. 57, no. 6, pp. 1652-1662, June. 2009.

[11] Y. Zhang and X. Da, "Construction of girth-eight QC-LDPC codes from arithmetic progression sequence with large column weight," *Electronics Letters*, vol. 51, no. 16, pp. 1257-1259, April 2015.

[12] J. Zhang and G. Zhang, "Deterministic girth-eight QC-LDPC codes with large column weight," *IEEE Communications Letters*, vol. 18, no. 4, pp. 656-659, April 2014.

[13] L. Wang, X. Zhang, F. Yu, *et al*., "QC-LDPC codes with girth eight based on independent row-column mapping sequence," *IEEE Communications Letters*, vol. 17, no. 11, pp. 2140-2143, Nov. 2013.

[14] Y. J. Ko and J. H. Kim, "Girth conditioning for construction of short block length irregular LDPC codes," *Electronics Letters*, vol. 40, no. 3, pp. 187-188, Feb. 2004.

[15] X. Hu, E. Eleftheriou, and D. M. Arnold, "Regular and irregular progressive edge-growth tanner graphs," *IEEE Trans. on Information Theory*, vol. 51, no. 1, pp. 386-398, Jan. 2005.

[16] S. T. Brink, G. Kramer, and A. Ashikhmin, "Design of low-density parity-check codes for modulation and detection," *IEEE Trans. on Communications*, vol. 52, no. 4, pp. 670-678, April 2004.

[17] M. Karimi and A. H. Banihashemi, "A message-passing algorithm for counting short cycles in a graph," *IEEE Information Workshop*, vol. 61, no. 2, pp. 1-17, Jan. 2010.

[18] Y. Mao and A. H. Banihashemi, "A heuristic search for good Low-density Parity-check codes at short block lengths," in *Proc. IEEE International Conf. on Communications*, Beijing, 2001, pp. 41-44.

[19] L. Yang, H. Liu, and C. J. R. Shi, "Cyclic elimination method to construct VLSI oriented LDPC codes," in *Proc. IEEE Vehicular Technology Conf.*, Dallas, 2005, pp. 522-526.

[20] T. Richardson and R. Urbanke, "The capacity of low density parity check codes under message passing decoding," *IEEE Trans. on Information Theory*, vol. 47, no. 2, pp. 599-618, Feb. 2001.

[21] J. G. Yuan, L. Xu, and Q. Tong, "A novel QC-LDPC code based on the finite field multiplicative group for optical communications," *Optoelectronics Letters*, vol. 9, pp. 287-380, Sep. 2013.

[22] J. Wang, G. Zhang, and Q. Zhou, "Coset-based QC-LDPC codes without small cycles," *IEEE Electronics Letters*, vol. 50, no. 22, pp. 1597-1598, Oct. 2014.

[23] L. J. Kong and Y. Xiao, "Design of good QC-LDPC codes without small girth in the *p*-plane," *Journal of Systems Engineering and Electronics*, vol. 22, no. 2, pp. 183-187, April 2011.

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