Nonlinear Analysis and Optimal Control of an Improved SIR Rumor Spreading Model

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Abstract — Based on the classic SIR model, this paper proposes a new rumor spreading model which considers the transition probability of nodes from S to I as a variable instead of a constant. This paper firstly studies the nonlinear phenomena in the new model, such as balance node, Hopf bifurcation, limit cycle. Secondly, this paper studies how to control the range of limit cycle. Finally, this paper studies the optimal control method, which can minimize the people who spread rumors.

Index Terms—Rumor spreading model, balance node, Hopf bifurcation, limit cycle, optimal control

I. INTRODUCTION

With the Internet popularizing, our life has been affected in every aspect and Internet rumors cause certain negative influence in our daily life. Based on the classic SIR model [1], the infected nodes have a constant probability to infect the susceptible nodes. However, we know that the spreading capability does not increase all the time. Capasso and Serio [2] proposed that the contagion capacity will probably decrease with a huge amount of infected nodes in 1973 when studying the cholera spreading. Based on this point, this paper proposes an improved SIR model, then uses a kind of nonlinear analysis on the improved model and studies the optimal control to decrease the rumor spreading.

The earliest researches on rumors spreading began with 1960s. Rumor spreading is similar to the virus spreading in the computer network starting with 1760 by Bernoulli. His paper studied on vaccinating to resist the variola. The research about the contagion math model started from early 20th century. Hamer and Ross [3], [4] make a huge contribution to establish the contagion math model. Kermack and McKendrick [5], [6] proposed SIR model when studying the plague contamination in London and proposed SIS model in 1932. The first classic rumor spreading model was proposed by Daley and Kendall [7], [8], called DK model. In DK model, the population was subdivided into S, I and R groups, assuming that the transition between and meets a certain mathematical probability distribution. Since then, Maki and Thomson [9], and Murray [10] have applied a mathematical model to study the rumors, and these studies mainly focused on the theoretical analysis.

This paper proposed an improved SIR model which is a piecewise function, aiming to change the transition probability from healthy nodes to spreading nodes. In Section II, this paper introduces the improved SIR model. In Section III, this paper discusses the stability of the balance nodes. In Section IV, this paper discusses the Hopf bifurcation and limit cycle of the model and in Section V, this paper introduces a controllable variable and discuss optimal control.

II. THE IMPROVED SIR MODEL

The number of users joining the network in unit time in classic SIR model is a constant which does not conform to the reality in social networks. Research has shown that the increase of users satisfies the logistic function

$$\frac{dS}{dt} = \gamma S(1 - \frac{S}{M}) - \alpha SI - \mu S$$
$$\frac{dI}{dt} = \alpha SI - \beta I - \mu I$$
$$\frac{dR}{dt} = \beta I - \mu R$$

(1)

S denotes the healthy node, I denotes the spreading node, R denotes the immune node. \(\alpha\) denotes the transition probability from healthy node to the spreading node when they are contacting. \(\beta\) denotes the transition probability from spreading node to the immune node when they are contacting. \(\mu\) denotes removing probability from the network in unit time. \(\gamma\) denotes the inherent increasing rate, \(k\) denotes the environment capacity and \(\gamma > \mu\).

Considering \(\alpha\) is a constant, the increasing number from healthy nodes to spreading nodes is linear with the increase of the spreading nodes. It does not meet the reality because the increasing number from healthy nodes to spreading nodes will be saturated with the increase of \(I\) based on the cholera spreading proposed by Capasso and Serio in 1973. Therefore, this paper uses a piecewise function \(T(I)\) instead of \(\alpha I\), where \(I_0\) is the spreading critical point.
\[
\begin{align*}
\frac{dS}{dt} & = \gamma S (1 - \frac{S}{k}) - ST(I) - \mu S \\
\frac{dI}{dt} & = ST(I) - \beta I - \mu I \\
\frac{dR}{dt} & = \beta I - \mu R
\end{align*}
\] (2)

where

\[ T(I) = \begin{cases} 
\alpha I, & I < I_0 \\
\alpha I_0, & I \geq I_0 
\end{cases} \] (3)

III. THE EXISTENCE AND STABILITY OF BALANCE NODES

At first, this paper calculates the balance nodes of the model. In order to search the balance, make the right of (2) equal to 0. Because the first two equations of (2) have none business with the third one, this paper obtains (4) and (5).

\[
\begin{align*}
\gamma S (1 - \frac{S}{k}) - \alpha SI - \mu S & = 0 & 0 \leq I \leq I_0 \\
\alpha SI - \beta I - \mu I & = 0 \\
\gamma S (1 - \frac{S}{k}) - \alpha SI_0 - \mu S & = 0 & I > I_0 \\
\alpha SI_0 - \beta I - \mu I & = 0
\end{align*}
\] (4) (5)

The Jacobian matrices of (4) and (5) are as follows.

\[
\begin{pmatrix}
\gamma - 2 \frac{\gamma S}{k} & -\alpha I & -\alpha S \\
\alpha I & \alpha S - \beta - \mu & 0 \\
\gamma - 2 \frac{\gamma S}{k} & -\alpha I_0 & -\beta - \mu
\end{pmatrix}
\]

Equation (4) has a non-trivial balance node and a non-rumor balance node [11]. There exists a non-trivial balance node \( E_0 = (0, 0) \), a non-rumor balance node \( E_0^* = (\frac{k(\gamma - \mu)}{\gamma}, 0) \), and a rumor spreading balance node \( E^* = (S^*, I^*) \), where \( S^* = \frac{\beta + \mu}{\alpha}, \quad I^* = \frac{\gamma - \mu}{\alpha} - \frac{\gamma(\beta + \mu)}{ka^2} \).

Equation (5) has the other balance node \( E_1 = (S_1, I_1) \), where \( S_1 = \frac{k(\gamma - \mu) - k\alpha I_0}{\gamma}, \quad I_1 = \frac{k\alpha I_0 (\gamma - \mu)}{\gamma(\beta + \mu)} - \frac{k\alpha^2 I_0^2}{\gamma(\beta + \mu)} \).

Set the basic reproductive rate \( R_0 = \frac{ak(\gamma - \mu)}{\gamma(\beta + \mu)} \). This paper gives the Corollary. 1.

**Corollary. 1.** Considering the model (2), \( R_0 = \frac{ak(\gamma - \mu)}{\gamma(\beta + \mu)} \), there exists the following conclusions.

(a) \( E_0^* \) is global asymptotic stable when \( R_0 < 1 \).

(b) \( E_0^* \) is global asymptotic stable when \( 1 \leq R_0 \leq 1 + \alpha I_0 \) and \( R_0 < \frac{ak}{\gamma} \).

(c) \( E_1 \) is global asymptotic stable when \( R_0 > 1 + \frac{k\alpha^2 I_0}{\gamma(\beta + \mu)}, \quad \alpha I_0 < \frac{1}{2} (\gamma - \mu) \) and \( \alpha I_0 < 2\sqrt{\alpha I_0 (\beta + \mu)} \).

**Fig. 1.** Phase diagram of \( I-S \) with the following parameters \( k = 1000, \alpha = 0.001, \gamma = 1, \mu = 0.5, \beta = 0.52, I_0 = 12.5 \).

**Fig. 1** meets the condition (a) of Corollary. 1, so the model (2) has a global asymptotic stable balance node \( E_0^* (500, 0) \).

**Fig. 2.** Phase diagram of \( I-S \) with the following parameters \( k = 1000, \alpha = 0.001, \gamma = 1, \mu = 0.288, \beta = 0.42, I_0 = 27 \).

**Fig. 2** meets the condition (b) of Corollary. 1, so the model (2) has a global asymptotic stable balance node \( E^* (708, 4) \).

**Fig. 3.** Phase diagram of \( I-S \) with the following parameters \( k = 20000, \alpha = 0.01, \gamma = 1, \mu = 0.1, \beta = 0.9, I_0 = 22.22 \).
Fig. 3 meets the condition (c) of Corollary 1, so the model (2) has a global asymptotic stable balance node.

IV. HOPF BIFURCATION

Form the Jacobian matrix of the balance nodes, it can be inferred that Hopf bifurcation may occur only in the neighbourhood of $E^*$.

Set $x = S - S^*$, $y = I - I^*$, so $E^*$ is translated to $(0,0)$.

Then the model can be described as (6).

\[
\begin{align*}
\frac{dx}{dt} &= a_1x + a_{12}y + f_1(x,y) \\
\frac{dy}{dt} &= a_2x + a_{22}y + f_2(x,y)
\end{align*}
\]

(6)

where

\[
\begin{align*}
a_{1} &= \gamma - \frac{2\gamma S^*}{k} - \alpha I^* - \mu \\
a_{12} &= \alpha S^* \\
a_{2} &= \alpha I^* \\
a_{22} &= \alpha S^* - \beta - \mu \\
f_1(x,y) &= -\frac{\gamma}{k} x^2 - \alpha xy \\
f_2(x,y) &= \alpha xy
\end{align*}
\]

(7)

Choose the appropriate parameters in order to make the trace of the Jacobian matrix be zero and the determinant of the Jacobian matrix be $\omega^2$, then eigenvalues can be described as $\pm \omega k$. Set $x = a_{12}y$, $y = ax - a_{11}y$, then the model can be described as (8).

\[
\begin{align*}
\frac{dx}{dt} &= -\omega y + \frac{1}{\omega} f_1(a_{12}y, ax - a_{11}y) - f_1(a_{12}y, ax - a_{11}y) \\
\frac{dy}{dt} &= ax + \frac{1}{a_{11}} f_1(a_{12}y, ax - a_{11}y)
\end{align*}
\]

(8)

Equation (8) is the same as (9).

\[
\begin{align*}
\frac{dx}{dt} &= -\omega y + F_1(x,y,\alpha) \\
\frac{dy}{dt} &= ax + F_2(x,y,\alpha)
\end{align*}
\]

(9)

Set the Lyapunov coefficient as (10).

\[
\sigma(\alpha) = \frac{1}{16} \left[ A_{\alpha} \right] + \frac{1}{16} \left[ \frac{1}{\alpha} \right] + \frac{1}{16} \left[ \frac{1}{\omega} \right] - \frac{1}{16} \left[ \frac{1}{\alpha} \right] + \frac{1}{16} \left[ \frac{1}{\omega} \right]
\]

(10)

where

\[
\begin{align*}
F_1(x,y,\alpha) &= a_{12} \omega (1 + \frac{1}{\alpha}) y^2 + [\frac{\gamma}{k} x^2 - a_{11} \alpha (1 + \frac{1}{\alpha})] y^2 \\
F_2(x,y,\alpha) &= a_{12} \omega (1 + \frac{1}{\alpha}) y^2 + [\frac{\gamma}{k} x^2 - a_{11} \alpha (1 + \frac{1}{\alpha})] y^2
\end{align*}
\]

(11)

By calculating it can be inferred (12).

\[
\sigma(\alpha) = \frac{1}{16} \left[ A_{\alpha} \right] + \frac{1}{16} \left[ \frac{1}{\alpha} \right] + \frac{1}{16} \left[ \frac{1}{\omega} \right] - \frac{1}{16} \left[ \frac{1}{\alpha} \right] + \frac{1}{16} \left[ \frac{1}{\omega} \right]
\]

(12)

Through analyzing Corollary 2 can be concluded.

**Corollary. 2.** $\alpha = \alpha_0$ is the bifurcate value, if the trace of the Jacobian matrix is zero and the determinant of the Jacobian matrix is $\omega^2$, and $\omega > 0$, $\sigma'(\alpha_0) \neq 0$. There will be a limit cycle in the neighbourhood of the balance node $E^*$.

![Fig. 4. The change of S, I, and R over time with the parameters (S,I,R)=(1.5,1.5,0.7), k=100,0.043, y=0.099, \mu=0.05, \beta=0.05, I_0=2.](image)

![Fig. 5. Three-dimensional stereogram about S, I, and R with the parameters (S,I,R)=(1.5,1.5,0.7), k=100, \alpha=0.043, y=0.099, \mu=0.05, \beta=0.05, I_0=2.](image)
Fig. 6. Phase diagram of $I-S$ with the following parameters $(S,I,R)=(1.5,1.5,0.7)$, $k=100$, $\alpha=0.043$, $\gamma=0.099$, $\mu=0.05$, $\beta=0.05$, $I_0=2$.

By calculating the parameters, it gets $S^*=2.2356$, $I^*=1.086$. In Fig. 4, it is clearly seen that the values of $S$ and $I$ are asymptotic stable to $2.3256$ and $1.086$. Fig. 5 is the three-dimensional stereogram about $S$, $I$, and $R$. It shows that the track is asymptotic stable to a node. In Fig. 6, it shows that the track is asymptotic stable to $(2.3256, 1.086)$ more clearly.

Then make $\alpha$ increase to $0.1$, the other parameters and original values of $S$, $I$, and $R$ remain unchanged. The simulation result is shown as Fig. 6.

Fig. 7. The change of $S$, $I$, and $R$ over time with the parameters $(S,I,R)=(1.5,1.5,0.7)$, $k=100$, $\alpha=0.1$, $\gamma=0.099$, $\mu=0.05$, $\beta=0.05$, $I_0=2$.

Fig. 8. Three-dimensional stereogram about $S$, $I$ and $R$ with parameters $(S,I,R)=(1.5,1.5,0.7)$, $k=100$, $\alpha=0.1$, $\gamma=0.099$, $\mu=0.05$, $\beta=0.05$, $I_0=2$.

Based on the parameters given in Fig. 7, through the calculation, we can get that the trace of the Jacobian matrix is zero and the determinant of the Jacobian matrix is $0.0048$, what’s more $\sigma'(0.1)=0.796$. According to Corollary 2, Hopf bifurcation will occur and the topological structure of track will change suddenly.

Calculate the coordinate $(1, 0.4801)$ of balance node $E^*$. From Fig. 7, it is intuitively seen that the three variables have stable periodic solution. From Fig. 8, it shows that the track is a limit cycle around the balance node $E^*$. From Fig. 9, it illustrates that the track is a limit cycle on the phase diagram of $I-S$.

Fig. 9. Phase diagram of $I-S$ with the following parameters $(S,I,R)=(1.5,1.5,0.7)$, $k=100$, $\alpha=0.1$, $\gamma=0.099$, $\mu=0.05$, $\beta=0.05$, $I_0=2$.

Then make $\alpha$ increase to $0.12$, the other parameters and original values of $S$, $I$, and $R$ remain unchanged. The numerical simulation is shown below:

Fig. 10. Three-dimensional stereogram about $S$, $I$, and $R$ with the parameters $(S,I,R)=(1.5,1.5,0.7)$, $k=100$, $\alpha=0.12$, $\gamma=0.099$, $\mu=0.05$, $\beta=0.05$, $I_0=2$.

Fig. 11. Phase diagram of $I-S$ with the parameters $(S,I,R)=(1.5,1.5,0.7)$, $k=100$, $\alpha=0.12$, $\gamma=0.099$, $\mu=0.05$, $\beta=0.05$, $I_0=2$.
According to the following parameters $k = 100$, $\alpha = 0.12$, $\gamma = 0.099$, $\mu = 0.05$, $\beta = 0.05$, $I_0 = 2$, calculate the coordinate $(1, 0.4801)$ of balance node $E^*$. From Fig. 10 and Fig. 11, it shows that the track is gradually stabilizing and finally be a limit cycle around the balance node $E^*$.

When the parameters are $k = 100$, $\gamma = 0.099$, $\beta = 0.05$, $\mu = 0.05$ and $I_0 = 2$. Through calculation we know that $\alpha = \alpha_0 = 0.1055$ is the bifurcation point. Then System (26),

\[ G_{101} = \frac{1}{2} \left( \frac{\partial^2 Q_1}{\partial y_1 \partial y_3} + \frac{\partial^2 Q_2}{\partial y_2 \partial y_3} + \frac{\partial^2 Q_3}{\partial y_1 \partial y_2} + \frac{\partial^2 Q_4}{\partial y_1 \partial y_3} + \frac{\partial^2 Q_5}{\partial y_2 \partial y_3} + \frac{\partial^2 Q_6}{\partial y_1 \partial y_2} \right) = 0 \]

(19)

\[ G_{101} = \frac{1}{2} \left( \frac{\partial^2 Q_1}{\partial y_1 \partial y_3} - \frac{\partial^2 Q_2}{\partial y_2 \partial y_3} + \frac{\partial^2 Q_3}{\partial y_1 \partial y_2} - \frac{\partial^2 Q_4}{\partial y_1 \partial y_3} + \frac{\partial^2 Q_5}{\partial y_2 \partial y_3} - \frac{\partial^2 Q_6}{\partial y_1 \partial y_2} \right) = 0 \]

(20)

\[ w_{11} = -\frac{1}{4\lambda_1(\alpha_0)} \left( \frac{\partial^2 Q_1}{\partial y_1^2} + \frac{\partial^2 Q_2}{\partial y_2^2} \right) = -0.001627 \]

(21)

\[ w_{20} = -\frac{1}{4(2\alpha_0(\alpha_0) - \lambda_2(\alpha_0))} \approx -0.150224 - 0.054784 \]

(22)

Then calculate the coefficient of curvature of limit cycle.

\[ \bar{\sigma}_1 = \text{Re} \left\{ \frac{g_{20} g_{11}}{g_{11}} \right\} + G_{101} w_{11} + G_{30} + G_{60} w_{20} \]

(24)

We design a square feedback controller

\[ U = k_2 (I - 0.4784) \]

add to the second equation of system (13), then we get a controlled system (25).

\[ \begin{aligned}
\frac{dS}{dt} &= 0.049S - 0.00099S^2 - 0.10055SI \\
\frac{dI}{dt} &= 0.10055SI - 0.1I \\
\frac{dR}{dt} &= 0.05I - 0.05R
\end{aligned} \]

(25)

By linear transformation like (14), System (25) can be described as (26).

\[ \begin{aligned}
\frac{dx}{dt} &= -0.0693 y + Q_1(x, y, z, \alpha_0) \\
\frac{dy}{dt} &= 0.0693 x + Q_2(x, y, z, \alpha_0) \\
\frac{dz}{dt} &= -0.05 z + Q_3(x, y, z, \alpha_0)
\end{aligned} \]

(15)

(26)

Then calculate the following Characteristic quantities.

\[ g_{11} = \frac{1}{4} \left( \frac{\partial^2 Q_1}{\partial y_1^2} + \frac{\partial^2 Q_2}{\partial y_2^2} + \frac{\partial^2 Q_3}{\partial y_1 \partial y_2} + \frac{\partial^2 Q_4}{\partial y_1 \partial y_3} + \frac{\partial^2 Q_5}{\partial y_2 \partial y_3} + \frac{\partial^2 Q_6}{\partial y_1 \partial y_2} \right) = 0.03484 - 0.0500275I \]

(18)
The error of them is 3%. when $\alpha=0.1, k_2=-0.000006$. Through formula (31), the calculated range of limit cycle is 1.3210. Through MATLAB we can get Fig. 13, in Fig. 13, the range of limit cycle is 1.3155. The error of them is 0.4%. when $\alpha=0.1, k_2=0.00004$. Through formula (31), the calculated range of limit cycle is 0.5835. Through MATLAB we can get Fig. 14, in Fig. 14, the range of limit cycle is 0.5855. The error of them is 0.35%.

In conclusion, controller $U = k_2 (I - 0.4784) \dot{I}$ can control the range of limit cycle availably, and the range can be calculated by (31). Fig. 15 is Phase diagram of $I-S$ with different values of $k_2^2$. 

---

From (24), (27), (28) and (29), we can get the coefficient of curvature of limit cycle in system (26).

$$\sigma_1 = \sigma_1 + \text{Re}(\psi) + \phi \tag{30}$$

Then calculate the range of limit cycle in system (26). For $\alpha < \alpha_0$ and $|\alpha-\alpha_0| \ll 1$, the range of limit cycle in system (26) can be described as (31).

$$Ra = \sqrt{\frac{e'(\alpha_0)}{e_1}(\alpha-\alpha_0)} = 45.42 \sqrt{\frac{0.10055 - \alpha}{1 + 58304.727 k_2^2}} \tag{31}$$

---

$\psi = \frac{w_{20}}{4}(C_{11}^2 - C_{22}^2) + \frac{w_{20}}{4}(C_{11}^2 + C_{22}^2)i$

$$+ \frac{8}{4\alpha_0}(C_{11}^2 + iC_{12}^2) - \frac{8}{4\alpha_0}(C_{11}^2 - iC_{11}^2) + \frac{8}{4\alpha_0}(C_{22}^1 - C_{22}^1) + \frac{8}{4\alpha_0}(C_{11}^1 + C_{22}^1)$$

$$+ \frac{s}{4\alpha_0}(C_{11}(\lambda_2^2(\alpha_0)i) + 2\alpha_0 i)(C_{11}^2 - C_{11}^1)$$

$$+ \frac{s}{4\alpha_0}(C_{11}^2 - C_{11}^1) + \frac{s}{4\alpha_0}(C_{22}^2 - C_{22}^1) - \frac{s}{4\alpha_0}(C_{11}^2 + C_{22}^1)$$

$$= 0 \tag{28}$$

$\phi = \frac{1}{8}(3D_{11} + D_{12} + D_{22} + 3D_{12})$

$$+ \frac{w_{11}}{2}(C_{11}^2 + C_{22}^2) + \lambda_2(\alpha_0)i + \frac{1}{4\alpha_0}(C_{11}^2 - C_{22}^1)$$

$$+ \frac{1}{8\alpha_0}(C_{12}^2 + C_{12}^2 - C_{22}^1 - C_{22}^1)$$

$$- \lambda_2(\alpha_0)i$$

$$= 0 \tag{29}$$

---

When $\alpha=0.1, k_2=0$, Through formula (31), the calculated range of limit cycle is 1.0652. Through MATLAB we can get Fig. 12, in Fig. 12, the range of limit cycle is 1.1045.
From Fig. 15, we can see the change of the range of limit cycle intuitively. We can increase or decrease the range of limit cycle by changing the value of $k_2$, and the range can be calculated by (31).

V. OPTIMAL CONTROL

The system above does not take any measures to control the rumor spreading. A way to reduce the rumor spreading is to take precautions against rumors, making the healthy people immune to rumors. In order to express the degree of precaution, this paper introduces a new controllable variable $u(t)$, $u(t) \in U_ad$. $U_ad = \{u(t): 0 \leq u(t) \leq 1, t \in [0, t_{final}]\}$ which denotes the number of vulnerable people transforming to the immune people in unit time [12].

Our target is to minimize the infected people and maximize the immune people through selecting the appropriate protect scope. In this way, our controllable target is to make the function: $J(u) = \int_0^T [I(t) + \frac{1}{2} \tau u^2(t)] dt$ reach the minimum value. $\tau$ denotes the protecting budget profile, $0 < \tau < \frac{\gamma S(1 - \frac{S}{I})}{\mu}$. The model after controlling can be described as (32).

$$
\frac{dS}{dt} = \gamma S(1 - \frac{S}{K}) - ST(I) - \mu S - uS
$$

$$
\frac{dI}{dt} = ST(I) - \beta I - \mu I
$$

$$
\frac{dR}{dt} = \beta I - \mu R + uS
$$

In order to search the minimum of the function, this paper defines the Hamiltonian function (33).

$$
H(t) = L(I,u) + \lambda_1(t) \frac{dS}{dt} + \lambda_2(t) \frac{dI}{dt} + \lambda_3(t) \frac{dR}{dt}
$$

$$
= I(t) + \frac{1}{2} \tau u^2(t) + \lambda_1(t) [\gamma S(1 - \frac{S}{K}) - ST(I) - \mu S - uS]
$$

$$
+ \lambda_2(t) [ST(I) - \beta I - \mu I] + \lambda_3(t) [\beta I - \mu R + uS]
$$

where $L(I,u) = I(t) + \tau u^2(t)/2$, $L(I,u)$ is the Lagrangian function. $I(t)$ denotes the number of $I$ in time $t$. $\tau u^2(t)/2$ denotes the consume of the system in order to reduce $I(t)$.

$\lambda_1(t)$ is an adjoint function as (34).

$$
\frac{d\lambda_1(t)}{dt} = -\lambda_1(t) \gamma + \frac{2}{K} S T(I) - \mu - u + \lambda_2(t) I + \lambda_3(t) u
$$

$$
\frac{d\lambda_2(t)}{dt} = \begin{cases} [2 \lambda_1(t) a S + \lambda_2(t) (a S - \mu) + \lambda_3(t) \beta] & I < I_0 \\ [-2 \lambda_1(t) a S + \lambda_2(t) (a S - \mu) + \lambda_3(t) \beta] & I > I_0 \\ \frac{\lambda_2(t)}{\mu} & I = I_0 \end{cases}
$$

Through optimum condition, it can be inferred (35).

$$
\frac{\partial H}{\partial u} = \tau u(t) - \lambda_1(t) S^* + \lambda_2(t) S^* = 0
$$

Equation (35) can be also described as (36).

$$
\begin{align*}
0 & \leq \frac{\partial H}{\partial u} < 0 \\
\frac{\partial H}{\partial u} &= \left( \frac{S^*}{\mu} - \frac{\lambda_1(t)}{\lambda_2(t)} \right) \\
\frac{\partial H}{\partial u} &= \frac{\partial H}{\partial u} = 0
\end{align*}
$$

This equation (36) can be also described as (37).

$$
\frac{\partial H}{\partial u} = \max \left\{ \min \left\{ \frac{\lambda_1(t)}{\lambda_2(t)} - \lambda_1(t) S^*, 1 \right\}, 0 \right\}
$$

Through the analysis above, this paper gives Corollary. 3 as follows.

**Corollary.** Considering the control, $S^*(t), I^*(t), R^*(t)$ are the optimal solution under the controllable variable $u^*(t)$, then there exists the adjoint variable $\lambda_1(t), \lambda_2(t), \lambda_3(t)$, which satisfy (34). The boundary condition is $\lambda_3(t_{final}) = 0, i = 1, 2, 3$. In this way, it can calculate (38) to obtain the optimal solution and the Hamiltonian function (39).

$$
\begin{align*}
\frac{dS^*}{dt} &= \gamma S^*(1 - \frac{S^*}{K}) - ST(I^*) - \mu S^* \\
&\quad - (\max \{\min \{\lambda_1(t) - \lambda_2(t) S^* / \tau, 1\}, 0\}) S^* \\
\frac{dI^*}{dt} &= \gamma S^* S - \beta I^* - \mu I^* \\
\frac{dR^*}{dt} &= \beta I^* - \mu R^* + (\max \{\min \{\lambda_1(t) - \lambda_2(t) S^* / \tau, 1\}, 0\}) S^*
\end{align*}
$$

In order to get the change trend of the controllable variable, this paper uses the Runge-Kutta [13] to simulate the system.

![Fig. 16. The number of $S$ before and after control with the parameters ($\kappa, \alpha, \gamma, \mu, \beta, \tau$) = (0.0001, 0.0001, 0.05, 0.052, 200) and $S(0) = 500$.](image)

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In Fig. 16, it can be clearly seen that the number of $S$ nodes decreases quickly to 0 after control while the number of $S$ nodes slowly increases then keeps invariant without control.

In Fig. 17, the number of $I$ nodes keeps a small rate to decrease after control while it keeps a huge rate to increase without control. In Fig. 18, it shows that the number of the $R$ nodes increases firstly, reaching to the maximum, then decreases after introducing the controllable variable.

Because of the precaution to the rumor, some healthy nodes transform to the immune nodes directly, leading the decrease of the healthy nodes and the increase of the immune nodes at first. Then, because of the decrease of the healthy nodes, the number of spreading nodes also decreases. When the number of immune nodes reaches the maximum, it will slowly decrease because the number of healthy nodes and immune nodes is close to 0, and there exists immune nodes removing from the network. To some extent, it can be concluded that the spreading of the rumors has controlled.

It is seen from the Fig. 19 that the controllable variable $u(t)$ first increases sharply, reaching the maximum quickly and then decreases at a slow speed. Through the change trend in the Fig. 19, the target function can reach the minimum value. Simulation results show that we can control rumor spreading better, and lower the resource cost.

VI. CONCLUSION

Based on the classic SIR model, this paper proposes a new rumor spreading model which considers the transition probability of nodes from $S$ to $I$ as a variable instead of a constant.

Firstly, this paper studies the nonlinear phenomena in the new model, such as balance node, Hopf bifurcation, limit cycle.

Secondly, this paper studies how to control the range of limit cycle.

Finally, this paper studies the optimal control method, which can minimize the people who spread rumors.

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