On the Full Diversity Property of A Space-Frequency Code Family with Multiple Carrier Frequency Offsets in Cooperative Communication Systems

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***Abstract*—Space-frequency (SF) coded orthogonal frequency division multiplexing (OFDM) systems have been recently proposed for cooperative communications to achieve both full cooperative and full multipath diversities without the time synchronization requirement. In this paper, we show that the full diversity property still holds for a family of SF codes (rotation-based SF codes) when there are multiple carrier frequency offsets (CFOs) from relay nodes under the condition that the absolute values of the normalized CFOs are less than 0.5. We then prove that this full diversity property can be preserved if we seek to reduce the receiver complexity by using a zero forcing (ZF) method to equalize the multiple CFOs, before applying maximum likelihood (ML) decoding. Furthermore, by exploiting the properties of SF codes, we show that a specific permuted version of this family of SF codes can still achieve full diversity even when the inter-carrier interference (ICI) matrix is singular. This is possible as long as the maximum absolute value of the normalized CFOs is not less than 0.5. However, in this case the ZF method cannot be directly used to reduce the decoding complexity. To avoid the necessity of jointly considering all the subcarriers, two suboptimal detection methods are proposed in which full diversity is achieved even for singular ICI matrix. All these imply that the SF coded OFDM system is robust to both timing errors and frequency offsets.**

***Index Terms*—Cooperative communications, diversity, ICI, multiple CFOs, OFDM, space-frequency codes**

I. INTRODUCTION

Due to the fact that orthogonal frequency division multiplexing (OFDM) systems are robust to timing errors, space-frequency (SF) coded OFDM cooperative systems have been proposed to achieve full asynchronous cooperative diversity, such as in [1]–[4]. However, it is well known that OFDM is sensitive to carrier frequency offset (CFO) that may lead to inter-carrier interference (ICI). For the conventional OFDM systems, only one CFO between transmitter and receiver exists. Thus if the CFO is accurately estimated, it can be easily compensated. However, in an SF coded OFDM cooperative system, since an SF code matrix is transmitted from various distributed nodes, there may exist multiple CFOs, which makes it difficult for the receiver to synchronize the signals from multiple relays at the same time.

The problem of multiple CFOs in cooperative communications has been studied recently in [5]–[9]. In [5], a subcarrier-wise Alamouti coded OFDM system is considered and a simplified zero-forcing (ZF) equalizer is applied to suppress the ICI. In [6], delay diversity is considered and a minimum mean square error (MMSE) decision feedback equalizer (DFE) is employed by the destination node, where the DFE filter coefficient calculation requires a matrix inversion per information symbol. In [9], computationally efficient MMSE and MMSE-DFE equalizations are proposed for the linear convolutively coded cooperative systems in the presence of multiple CFOs, where the linear convolutively space-time codes [10] can achieve the full asynchronous cooperative diversity under any delay profile. Different from the methods in [6], the equalization methods proposed in [9] do not need to inverse a matrix per information symbol. In [7] and [8], some effective and efficient signal detection methods are proposed for an SF coded cooperative communication system in the presence of multiple CFOs, where rotation-based SF codes [11] are considered. Compared with codes used in [5], [6], [9], these SF codes are powerful in the sense that they can achieve both full cooperative diversity as well as full multipath diversity, and their rate is always equal to one regardless of the number of transmit antenna. Generally speaking, all of the above methods adopt the equalization techniques to deal with the multiple CFOs problem. However, it is not known if these equalization techniques can still guarantee full diversity.

In this paper, we consider the SF codes proposed in [11] for MIMO-OFDM systems. Above all, we show that the full diversity property still holds for this family of SF codes when there are multiple CFOs from relay

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nodes under the condition that the absolute values of the normalized CFOs are less than 0.5. The key idea for the proof is to treat the ICI terms due to CFOs as a part of an SF code matrix. It turns out that as long as the original SF code achieves the full diversity (both spatial and multipath diversities) and the absolute values of normalized CFOs are less than 0.5, the new (virtual) code after absorbing the CFOs/ICI into the original SF code maintains the full diversity. The above full diversity property is based on the maximum likelihood (ML) decoding across all the subcarriers of the OFDM system, which may have a high complexity. To overcome this difficulty, we further study an SF coded OFDM system where the ICI matrix is first equalized by a zero forcing (ZF) method, followed by ML decoding for the SF codes (we call this method the ZF-ML method). The complexity of this ZF-ML detection method is much reduced compared to the complete ML method described above, and we prove that the ZF-ML method still achieves the same diversity order as the case without CFOs.

We also address the more challenging case when the absolute values of the normalized CFOs are not smaller than 0.5. In this case we cannot guarantee the ICI matrix being nonsingular, as well as the full diversity property of the ML and ZF-ML method described above. To enhance the robustness of these SF codes to multiple CFOs, we devise a specific permuted (interleaved) SF codes that can still achieve full diversity even when the ICI matrix is singular. This full diversity property is still based on joint consideration of all the subcarriers of the OFDM system whose decoding complexity is high. To tackle this problem, we then propose two suboptimal detection methods with different tradeoff between efficiency and complexity, namely the ZF-ML-Zn method and the ZF-ML-PIC method. Both of these two methods can still achieve full diversity when the ICI matrix is singular. All these imply that such an SF coded OFDM cooperative system is robust to both timing errors and frequency offsets from the relay nodes.

The remaining of this paper is organized as follows. In Section II, the system model is described. In Section III, the structure of the SF codes in [11] is reviewed. In Section IV, the effect of multiple CFOs on the SF codes is analyzed. The next section shows that a specific permutation can improve the robustness of these SF codes to multiple CFOs. In Section VI, some simulations are presented to verify the theoretical results, and conclusions are given in Section VII. Throughout this paper, full channel knowledge including CFOs at the destination node, is assumed.

Some Notations: We use $\mathbf{A}(l,k)$ to denote the (l,k)th entry of $\mathbf{A}$, and $\mathbf{x}(k)$ to denote the kth entry of vector $\mathbf{x}$. Superscripts $^{\mathcal{T}}$, *, and $^{\mathcal{H}}$ stand for transpose, conjugate, and Hermitian, respectively. $\mathrm{E}[x]$ represents the expectation of variable x. Integer ceiling and floor are denoted by $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$, respectively. $\mathbf{I}_N$ represents the $N\times N$ identity matrix . The $N\times 1$ all zero and all one vectors are denoted by $\mathbf{0}_N$ and $\mathbf{1}_N$, respectively. $\mathbf{0}_{N\times M}$ stands for an $N\times M$ all zero matrix. $\mathrm{diag}(d_0,\ldots,d_{N-1})$ denotes an $N\times N$ diagonal matrix with diagonal scalar entries $d_0,\ldots,d_{N-1}$, and an $NQ\times NR$ block diagonal matrix with diagonal $Q\times R$ matrices $\mathbf{D}_0,\ldots,\mathbf{D}_{N-1}$ is denoted by $\mathrm{diag}(\mathbf{D}_0,\ldots,\mathbf{D}_{N-1})$. $\mathbf{F}_N$ is the $N\times N$ normalized FFT matrix. The Kronecker and Hadamard product are denoted by $\otimes$ and $\odot$, respectively. $(\,)_N$ means modular operation. The cardinality of a set $\mathcal{A}$ is denoted by $|\mathcal{A}|$.

II. System Model

A. Cooperative Protocol

Fig. 1 shows the cooperative communication system we use in this paper, which includes one source node, one destination node, and a number of relay nodes in the middle. In the first phase, the source node S broadcasts the information while the relays receive the same information. In the second phase, the M_t relays, which have detected the received information symbols correctly, will help the source to transmit. The detected symbols are parsed into blocks of size N and N is also the number of subcarriers in one OFDM symbol. Then, the bth block, $b=0,1,\ldots,$ is encoded to an SF code matrix $\mathbf{C}$ in a distributed fashion [2]. Finally, the mth, $1\le m\le M_t$, relay transmits the $(m-1)$th column of the code matrix, denoted as $\mathbf{c}_m$, by the standard OFDM technology. Hence, this is a *decode-and-forward* (DF) protocol. We assume that each node has only one transmit/receive antenna. Another DF protocol for these SF codes can be found in [12], to which our major conclusions obtained in this paper are still applicable. Although, in a cooperative communication system, each column of an SF code corresponds to one relay, to be consistent with the notation used in the SF coding literature, we will use the term *antenna* instead of *relay*

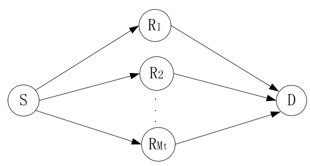

Fig. 1. System structure

B. Channel Model

The channel impulse response from the mth relay to the destination node is denoted as $h_m(\tau)=\sum_{l=0}^{L_m-1}\alpha_m(l)\delta(\tau-\tau_m(l))$, where L_m is the number of multipaths of the link from the mth relay to the destination node. The complex amplitude and delay for the lth multipath of the mth relay are $\alpha_m(l)$ and $\tau_m(l)$, respectively, where $\alpha_m(l)$ is a zero mean complex Gaussian variable with power $\mathrm{E}[|\alpha_m(l)|^2]=\sigma_m^2(l)$. The power of each link is normalized such that $\sum_{l=0}^{L_m-1}\sigma_m^2(l)=1$ for $m=1,2,\ldots,M_t$. We further assume that the delays for the relays are rounded to the sampling positions. The channel taps $\alpha_m(l)$ are assumed independent from each other for different m and l. The frequency response of the link between the mth relay and the destination node, $\mathrm{H}_m = [H_m(0), \ldots, H_m(N-1)]^{\mathcal{T}}$, is given by

$$\mathrm{H}_m = \hat{\mathbf{F}}_m \mathbf{h}_m, \quad (1)$$

where $\mathbf{h}_m = [\alpha_m(0), \ldots, \alpha_m(L_m - 1)]^{\mathcal{T}}$ and $\hat{\mathbf{F}}_m = [\mathbf{f}^{\tau_m(0)}, \ldots, \mathbf{f}^{\tau_m(L_m-1)}]$. The column vector $\mathbf{f}^{\tau_m(l)} = [1, \zeta^{\tau_m(l)}, \ldots, \zeta^{(N-1)\tau_m(l)}]^{\mathcal{T}}$, where $\zeta = \exp(-j\frac{2\pi}{T})$ and T is the duration of an OFDM symbol.

III. Structure of Space-Frequency Codes

The SF code for the cooperative communication system we consider is based on the codes in [11], [13], [14] and [2]. In this section we briefly review the structure and some properties of this SF code which will be utilized in the following sections. We adopt the code structure proposed in [11].

Each SF code $\mathbf{C}$ is an $N \times M_t$ matrix and is mapped from an $N' \times 1$ information symbol vector $\mathbf{s}$, where M_t is the number of transmit antennas and $N' \le N$. For the coding strategy proposed in [11], each SF code matrix $\mathbf{C}$ is a concatenation of some matrices $\mathbf{G}_p$ with the form

$$\mathbf{C} = \begin{bmatrix} \mathbf{G}_1^{\mathcal{T}} & \mathbf{G}_2^{\mathcal{T}} & \ldots & \mathbf{G}_P^{\mathcal{T}} & \mathbf{0}_{(N-N')\times M_t}^{\mathcal{T}} \end{bmatrix}^{\mathcal{T}}, \quad (2)$$

where $P = \lfloor N/(\Gamma M_t) \rfloor$, $N' = P\Gamma M_t$, each matrix $\mathbf{G}_p$, $1 \le p \le P$, is of size ΓM_t by M_t, and Γ is a coding parameter related to the achievable diversity order for the code. The zero padding matrix $\mathbf{0}_{(N-N')\times M_t}$ is used if the number of subcarriers N is not an integer multiple of ΓM_t. In the remainder of this paper, without loss of generality, we assume that N is an integer multiple of ΓM_t, i.e., $N = P\Gamma M_t$. Each matrix $\mathbf{G}_p$, $1 \le p \le P$, has the same structure given by

$$\mathbf{G}_p = \sqrt{M_t}\mathrm{diag}(\mathbf{x}_1^p, \mathbf{x}_2^p, \ldots, \mathbf{x}_{M_t}^p), \quad (3)$$

where $\mathbf{x}_m^p = [x_{(m-1)\Gamma+1}^p, \ldots, x_{m\Gamma}^p]^{\mathcal{T}}$ for $1 \le m \le M_t$, and all x_k^p, $1 \le k \le M_t\Gamma$, are mapped from an information subvector $\mathbf{s}^p = [s_1^p, \ldots, s_{\Gamma M_t}^p]^{\mathcal{T}}$ by a linear transform $[\mathbf{x}_1^{p\mathcal{T}}, \ldots, \mathbf{x}_{M_t}^{p\ \mathcal{T}}]^{\mathcal{T}} = \mathbf{\Theta}\mathbf{s}^p$, where s_l^p is $((p-1)\Gamma M_t + l - 1)$th entry of $\mathbf{s}$ for $1 \le l \le \Gamma M_t$ and $\mathbf{\Theta}$ is the $M_t\Gamma \times M_t\Gamma$ linear transformation matrix. The details regarding to the construction of $\mathbf{\Theta}$ can be found in [11]. The energy constraint is $\mathrm{E}\left[\sum_{k=1}^{\Gamma M_t} |x_k|^2\right] = \Gamma M_t$. The diversity order achieved by these SF codes is $M_r \sum_{m=1}^{M_t} \min(\Gamma, L_m)$ [2], [3], [11], where M_r is the number of receive antennas. So if $\Gamma \ge \max_m(L_m)$, full diversity order $(\sum_{m=1}^{M_t} L_m)M_r$ can be achieved.

One property of the above SF codes is that each subcarrier is only used by one transmit antenna. From (2) and (3), the $(m-1)$th, $1 \le m \le M_t$, column of $\mathbf{C}$, denoted as $\mathbf{c}_m$, can be written as

$$\mathbf{c}_m = \mathbf{P}_m \mathbf{c}_m. \quad (4)$$

where $\mathbf{P}_m$ is a diagonal *selection* matrix corresponding to the $(m-1)$th column of $\mathbf{C}$ given by

$$\mathbf{P}_m(l, l') = \begin{cases} 1, & \text{if } l = l' = (t-1)\Gamma M_t + (m-1)\Gamma + i \\ 0, & \text{else} \end{cases}, \quad (5)$$

where $0 \le l, l' \le N-1$, $0 \le i \le \Gamma - 1$ and $1 \le t \le P$. Based on (5) we also have

$$\mathbf{P}_m \mathbf{P}_{m^\dagger} = \begin{cases} \mathbf{P}_m, & \text{if } m = m^\dagger \\ \mathbf{0}_{N\times N}, & \text{otherwise} \end{cases}. \quad (6)$$

IV. Effect of Multiple CFOs on the SF Codes

A. Receive Signal Model

Let b, $b = 0, 1, 2, \ldots$, denote the OFDM symbol index. Then at the destination node, after standard steps, the bth received OFDM symbol $\mathbf{z}^b$ in the frequency domain is given by

$$\mathbf{z}^b = \sqrt{\frac{\rho}{M_t}} \sum_{m=1}^{M_t} e^{j\theta_{\varepsilon_m}^b} \mathbf{U}_{\varepsilon_m} \mathrm{diag}(\mathrm{H}_m)\mathbf{c}_m + \mathbf{w}, \quad (7)$$

where $\mathbf{w}$ is an $N \times 1$ noise vector with each entry being a zero mean unit variance complex Gaussian random variable and ρ stands for the signal-to-noise ratio (SNR) at the destination node. Let Δf_m be the CFO between the mth relay and the destination node. Then $\varepsilon_m = \Delta f_m T$ is its normalized value by OFDM symbol duration T. In (7), $\theta_{\varepsilon_m}^b = 2\pi\varepsilon_m(bN + bL_{cp} + L_{cp})/N + \theta_{0,m}$ where L_{cp} is the length of cyclic prefix, $2\pi\varepsilon_m(bN + bL_{cp} + L_{cp})/N$ is the phase rotation of the bth OFDM symbol transmitted from the mth relay induced by CFO Δf_m and $\theta_{0,m}$ is the phase rotation between the phase of the destination node local oscillator and the carrier phase of the mth relay at the start of the received signal. $\mathbf{U}_{\varepsilon_m}$ is the $N \times N$ ICI matrix induced by ε_m and is given by

$$\mathbf{U}_{\varepsilon_m} = \mathbf{F}_N \mathbf{\Omega}_{\varepsilon_m} \mathbf{F}_N^{\mathcal{H}}, \quad (8)$$

where $\mathbf{\Omega}_{\varepsilon_m} = \mathrm{diag}(1, e^{j2\pi\varepsilon_m/N}, \ldots, e^{j2\pi\varepsilon_m(N-1)/N})$.

Substituting (4) into (7), we have

$$\begin{aligned} \mathbf{z}^b &= \sqrt{\frac{\rho}{M_t}} \sum_{m=1}^{M_t} e^{j\theta_{\varepsilon_m}^b} \mathbf{U}_{\varepsilon_m} \mathrm{diag}(\mathrm{H}_m)\mathbf{P}_m\mathbf{c}_m + \mathbf{w} \\ &= \sqrt{\frac{\rho}{M_t}} \sum_{m=1}^{M_t} e^{j\theta_{\varepsilon_m}^b} \mathbf{U}_{\varepsilon_m} \mathbf{P}_m \mathrm{diag}(\mathrm{H}_m)\mathbf{c}_m + \mathbf{w}, \quad (9) \end{aligned}$$

where the identity $\mathrm{diag}(\mathrm{H}_m)\mathbf{P}_m = \mathbf{P}_m\mathrm{diag}(\mathrm{H}_m)$ is applied since $\mathrm{diag}(\mathrm{H}_m)$ and $\mathbf{P}_m$ are diagonal matrices. Based on (6), $\mathbf{P}_m\mathrm{diag}(\mathrm{H}_m)\mathbf{c}_m$ can be expressed by $\mathbf{P}_m \sum_{l=1}^{M_t} \mathbf{P}_l \mathrm{diag}(\mathrm{H}_l)\mathbf{c}_l$. From (9) we get

$$\begin{aligned} \mathbf{z}^b &= \sqrt{\frac{\rho}{M_t}} \sum_{m=1}^{M_t} e^{j\theta_{\varepsilon_m}^b} \mathbf{U}_{\varepsilon_m} \left(\mathbf{P}_m \sum_{l=1}^{M_t} \mathbf{P}_l \mathrm{diag}(\mathrm{H}_l)\mathbf{c}_l \right) + \mathbf{w} \\ &= \sqrt{\frac{\rho}{M_t}} \mathbf{U}^b \sum_{m=1}^{M_t} \mathrm{diag}(\mathrm{H}_m)\mathbf{c}_m + \mathbf{w}, \quad (10) \end{aligned}$$

where

$$\mathbf{U}^b = \sum_{m=1}^{M_t} e^{j\theta_{\varepsilon_m}^b} \mathbf{U}_{\varepsilon_m} \mathbf{P}_m. \quad (11)$$

So we can see that due to the property of the SF codes, i.e., each subcarrier is only used by one transmit antenna, the effect of ICI matrix $\mathbf{U}_{\varepsilon_m}$, $1 \le m \le M_t$, is incorporated into the matrix $\mathbf{U}^b$. Substituting (1) into (10), we obtain

$$\begin{aligned}\mathbf{z}^b &= \sqrt{\frac{\rho}{M_t}}\mathbf{U}^b \sum_{m=1}^{M_t} \mathrm{diag}(\hat{\mathbf{F}}_m \mathbf{h}_m)\mathbf{c}_m + \mathbf{w} \\ &= \sqrt{\frac{\rho}{M_t}}\mathbf{U}^b \sum_{m=1}^{M_t} [\mathbf{c}_m \odot \mathbf{f}^{\tau_m(0)}, \ldots, \mathbf{c}_m \odot \mathbf{f}^{\tau_m(L_m-1)}]\mathbf{h}_m \\ &\quad +\mathbf{w}. \end{aligned} \tag{12}$$

Assuming that $L = \max_m(L_m)$, we define $N \times M_t$ matrix $\mathbf{J}_l$ as

$$\mathbf{J}_l \triangleq [\mathbf{f}^{\tau_1(l)}, \mathbf{f}^{\tau_2(l)}, \ldots, \mathbf{f}^{\tau_{M_t}(l)}], \tag{13}$$

for $0 \le l \le L-1$, and $M_t L \times 1$ vector $\mathbf{h} \triangleq [\alpha_1(0), \ldots, \alpha_{M_t}(0), \ldots, \alpha_1(L-1), \ldots, \alpha_{M_t}(L-1)]$, where $\alpha_m(l) = 0$ and $\tau_m(l) = 0$, if $l \ge L_m$ for $1 \le m \le M_t$. By further defining $N \times M_t L$ matrix $\mathbf{X}$ as

$$\mathbf{X} = [\mathbf{J}_0 \odot \mathbf{C}, \mathbf{J}_1 \odot \mathbf{C}, \ldots, \mathbf{J}_{L-1} \odot \mathbf{C}], \tag{14}$$

the received signal $\mathbf{z}^b$ can be written as:

$$\mathbf{z}^b = \sqrt{\frac{\rho}{M_t}}\mathbf{U}^b \mathbf{X}\mathbf{h} + \mathbf{w}. \tag{15}$$

B. Diversity Analysis of SF Codes with Multiple CFOs

It is not hard to see that this signal model (15) is a standard SF coded MIMO-OFDM receive signal model which has been examined in [11], [2] and [3]. According to the *diversity (rank) criterion* of SF codes design, clearly in the presence of multiple CFOs, the achieved diversity order of SF codes is equal to the minimum rank of the matrix $\mathbf{U}^b(\mathbf{X}-\hat{\mathbf{X}})\mathbf{\Lambda}(\mathbf{X}-\hat{\mathbf{X}})^{\mathcal{H}}\mathbf{U}^{b\mathcal{H}}$ for any distinct $\mathbf{C}$ and $\hat{\mathbf{C}}$, where $\mathbf{\Lambda} = \mathrm{E}[\mathbf{h}\mathbf{h}^{\mathcal{H}}]$. Furthermore, we have the inequality

$$\begin{aligned} &\mathrm{rank}\left(\mathbf{U}^b(\mathbf{X}-\hat{\mathbf{X}})\mathbf{\Lambda}(\mathbf{X}-\hat{\mathbf{X}})^{\mathcal{H}}\mathbf{U}^{b\mathcal{H}}\right) \\ \le\ &\mathrm{rank}\left((\mathbf{X}-\hat{\mathbf{X}})\mathbf{\Lambda}(\mathbf{X}-\hat{\mathbf{X}})^{\mathcal{H}}\right). \end{aligned} \tag{16}$$

From (16) we can conclude that the achieved diversity order by this SF code with multiple CFOs can only be less than or equal to that without CFOs. On the other hand, when $\mathrm{rank}(\mathbf{U}^b) = N$, the equality in (16) holds. This means that the achieved diversity order is the same as the case without CFOs.

As multiple CFOs affect the SF codes through the matrix $\mathbf{U}^b$, let us investigate it in details. Substituting (8) into (11), we obtain

$$\begin{aligned}\mathbf{U}^b &= \sum_{m=1}^{M_t} e^{j\theta^b_{\varepsilon_m}} \mathbf{F}_N \mathbf{\Omega}_{\varepsilon_m} \mathbf{F}_N^{\mathcal{H}} \mathbf{P}_m \\ &= \mathbf{F}_N \sum_{m=1}^{M_t} \mathbf{\Omega}_{\varepsilon_m} \mathbf{F}_N^{\mathcal{H}} \mathbf{P}_m \left(\sum_{i=1}^{M_t} e^{j\theta^b_{\varepsilon_i}} \mathbf{P}_i\right) \\ &= N^{-\frac{1}{2}} \mathbf{F}_N \mathbf{V} \mathbf{P}^b, \end{aligned} \tag{17}$$

where $\mathbf{V} = N^{\frac{1}{2}} \sum_{m=1}^{M_t} \mathbf{\Omega}_{\varepsilon_m} \mathbf{F}_N^{\mathcal{H}} \mathbf{P}_m$, $\mathbf{P}^b = \sum_{i=1}^{M_t} e^{j\theta^b_{\varepsilon_i}} \mathbf{P}_i$, and the second equality follows from (6). Here it is easy to verify that $\mathbf{P}^b$ is a diagonal and unitary matrix. Due to the fact that $\mathbf{F}_N$ is a unitary matrix, we have $\mathrm{rank}(\mathbf{U}^b) = \mathrm{rank}(N^{-\frac{1}{2}}\mathbf{F}_N\mathbf{V}\mathbf{P}^b) = \mathrm{rank}(\mathbf{V})$. From the expression of $\mathbf{\Omega}_{\varepsilon_m}$, we can see that the matrix $N^{\frac{1}{2}}\mathbf{\Omega}_{\varepsilon_m}\mathbf{F}_N^{\mathcal{H}}$, $1 \le m \le M_t$, is a Vandermonde matrix [15] and its kth column, denoted by $\mathbf{q}^k_{\varepsilon_m}$, has the form

$$\mathbf{q}^k_{\varepsilon_m} = [1, e^{j2\pi(\varepsilon_m+k)/N}, e^{j2\pi(\varepsilon_m+k)2/N}, \ldots, e^{j2\pi(\varepsilon_m+k)(N-1)/N}]^{\mathcal{T}}. \tag{18}$$

As $\mathbf{P}_m$ defined in (5) is just a *selection* matrix, according to its property shown by (6), it is obvious that the kth column of $\mathbf{V}$, denoted by $\mathbf{v}_k$, is just the kth column of the matrix $N^{\frac{1}{2}}\mathbf{\Omega}_{\varepsilon_{m_k}}\mathbf{F}_N^{\mathcal{H}}$. Here we assume that the kth subcarrier is used by m_kth transmit antenna for $0 \le k \le N-1$ and $1 \le m_k \le M_t$. Therefore, we have

$$\mathbf{v}_k = \mathbf{q}^k_{\varepsilon_{m_k}} = [1, e^{j2\pi(\varepsilon_{m_k}+k)/N}, \ldots, e^{j2\pi(\varepsilon_{m_k}+k)(N-1)/N}]^{\mathcal{T}}. \tag{19}$$

It is clear that $\mathbf{V}$ is also a Vandermonde matrix. Thus the determinant of $\mathbf{V}$ is calculated by

$$\begin{aligned}\det(\mathbf{V}) &= \prod_{0\le i<l\le N-1} \left(e^{j2\pi(\varepsilon_{m_l}+l)/N} - e^{j2\pi(\varepsilon_{m_i}+i)/N}\right) \\ &= \prod_{0\le i<l\le N-1} e^{j2\pi(\varepsilon_{m_i}+i)/N}\left(e^{j2\pi(l-i+\varepsilon_{m_l}-\varepsilon_{m_i})/N} - 1\right). \end{aligned} \tag{20}$$

From (20), it is noted that $\det(\mathbf{V}) = 0$ if and only if we can find a pair of integers i and l such that $e^{j2\pi(l-i+\varepsilon_{m_l}-\varepsilon_{m_i})/N} - 1 = 0$ for $0 \le i < l \le N-1$. Since $2\pi(l-i+\varepsilon_{m_l}-\varepsilon_{m_i})/N = 2t\pi \Leftrightarrow \varepsilon_{m_l}-\varepsilon_{m_i} = tN+i-l$ for t is an integer, finally we get

$$\det(\mathbf{V}) = 0 \Leftrightarrow \varepsilon_{m_l}-\varepsilon_{m_i} = tN+i-l, \tag{21}$$

where $0 \le i < l \le N-1$ and all of t, i and l are integers. We now have the following theorem.

Theorem 1: If the absolute values of normalized CFOs ε_m, $1 \le m \le M_t$, are all less than 0.5, then the diversity order of the SF codes described in Section III is the same as the case without CFOs.

Proof: If $|\varepsilon_m| < 0.5$ for all m, we can get $-1 < \varepsilon_{m_l}-\varepsilon_{m_i} < 1$. On the other hand, from $0 \le i < l \le N-1$, we have $1-N \le i-l \le -1$. Thus, for (21) it is found that $\varepsilon_{m_l}-\varepsilon_{m_i}$ can only be integers except the points which are integer multiple of N. So if $|\varepsilon_m| < 0.5$ for all m, condition (21) cannot be satisfied and $\det(\mathbf{V}) \ne 0$, which implies that the matrix $\mathbf{U}^b$ has full rank and therefore the equality in (16) holds. This means that the achieved diversity is not affected by CFOs. ■

C. A Counterexample when $\varepsilon_{max} \ge 0.5$

On the other hand, if $\varepsilon_{Max} \ge 0.5$, the maximum achieved diversity order of the SF codes may be less than $M_t\Gamma$, although $L_m \ge \Gamma$ for $1 \le m \le M_t$. Here ε_{Max} is the maximum value of $|\varepsilon_m|$ for $1 \le m \le M_t$. The following is a counterexample corresponding to $\varepsilon_{m_l}-\varepsilon_{m_i} = -1$ which is possible if $\varepsilon_{Max} \ge 0.5$.

We define some additional notations. Denote the $\mathbf{X}_p$, $1 \le p \le P$, as an $M_t\Gamma \times M_t L$ submatrix of $\mathbf{X}$ which contains all the rows of $\mathbf{X}$ in (14) related to $\mathbf{G}_p$ given by (3). According to the structure of $\mathbf{X}$, $\mathbf{X}_p$ is just the $(p-1)$th $M_t\Gamma \times M_t L$ submatrix of $\mathbf{X}$. Due to the rows of $\mathbf{X}$ contained by $\mathbf{X}_p$, we also define $\bar{\mathbf{V}}_p$, $1 \le p \le P$, as the $(p-1)$th

$N\times M_t\Gamma$ submatrix of $\mathbf{V}$ and $\bar{\mathbf{P}}^b_p$, $1 \le p \le P$, as the $(p-1)$th main diagonal $M_t\Gamma\times M_t\Gamma$ matrix of $\mathbf{P}^b$, respectively. Then according to block matrix multiplication, we have

$$\mathbf{V}\mathbf{P}^b\mathbf{X} = \sum_{p=1}^{P} \bar{\mathbf{V}}_p\bar{\mathbf{P}}^b_p\mathbf{X}_p. \tag{22}$$

Suppose that $\mathbf{C}$ and $\hat{\mathbf{C}}$ are two distinct SF code matrices which are constructed from $\mathbf{G}_1, \ldots, \mathbf{G}_P$ and $\hat{\mathbf{G}}_1, \ldots, \hat{\mathbf{G}}_P$, respectively. Without loss of generality, we can consider the case that $\mathbf{G}_p = \hat{\mathbf{G}}_p$ for $p > 1$. Then the related difference matrices $\Delta\mathbf{X}_p = \mathbf{X}_p - \hat{\mathbf{X}}_p$ are all zero matrices for $2 \le p \le P$. So substituting this fact into (22), we have $\mathbf{V}\mathbf{P}^b(\mathbf{X} - \hat{\mathbf{X}}) = \bar{\mathbf{V}}_1\bar{\mathbf{P}}^b_1\Delta\mathbf{X}_1$.

From (2) and (5), we obtain $m_k = \lfloor k/\Gamma\rfloor + 1$ for $0 \le k \le M_t\Gamma - 1$. Thus according to (19), $\bar{\mathbf{V}}_1$ has the form

$$\begin{aligned}\bar{\mathbf{V}}_1 = [&\mathbf{q}^0_{\varepsilon_1}, \ldots, \mathbf{q}^{\Gamma-1}_{\varepsilon_1}, \mathbf{q}^{\Gamma}_{\varepsilon_2}, \ldots, \mathbf{q}^{2\Gamma-1}_{\varepsilon_2}, \ldots, \mathbf{q}^{(m^\dagger-1)\Gamma}_{\varepsilon_{m^\dagger}}, \ldots,\\ &\mathbf{q}^{m^\dagger\Gamma-1}_{\varepsilon_{m^\dagger}}, \mathbf{q}^{m^\dagger\Gamma}_{\varepsilon_{m^\dagger+1}}, \ldots, \mathbf{q}^{(m^\dagger+1)\Gamma-1}_{\varepsilon_{m^\dagger+1}}, \ldots, \mathbf{q}^{(M_t-1)\Gamma}_{\varepsilon_{M_t}}, \ldots,\\ &\mathbf{q}^{M_t\Gamma-1}_{\varepsilon_{M_t}}],\end{aligned} \tag{23}$$

where $1 \le m^\dagger \le M_t - 1$. If $\varepsilon_{Max} \ge 0.5$, it is possible that $\varepsilon_{m_{m^\dagger\Gamma}} - \varepsilon_{m_{(m^\dagger\Gamma-1)}} = \varepsilon_{m^\dagger+1} - \varepsilon_{m^\dagger} = -1$. Therefore condition (21) is satisfied corresponding to the case that $l = i + 1$ and $t = 0$. As a consequence, $\mathbf{V}$ is no longer a nonsingular matrix. Actually, from (19), we have $\mathbf{v}_{(m^\dagger\Gamma-1)} = \mathbf{q}^{m^\dagger\Gamma-1}_{\varepsilon_{m^\dagger}} = \mathbf{v}_{m^\dagger\Gamma} = \mathbf{q}^{m^\dagger\Gamma}_{\varepsilon_{m^\dagger+1}}$. As $\mathbf{v}_{(m^\dagger\Gamma-1)}$ and $\mathbf{v}_{m^\dagger\Gamma}$ are also two consecutive columns of $\bar{\mathbf{V}}_1$ in (23), $\bar{\mathbf{V}}_1$ is not of full column rank $M_t\Gamma$. For example, the structure of $\mathbf{V}$ for $M_t = 2$, $N = 8$ and $\Gamma = 2$ is given by (24) at the top of next page. By substituting $\varepsilon_2 = \varepsilon_1 - 1$ into (24), we can clearly see that in this case both $\bar{\mathbf{V}}_1$, which contains the first 4 columns of $\mathbf{V}$, and $\bar{\mathbf{V}}_2$, which contains the last 4 columns of $\mathbf{V}$, have no full column rank 4. Therefore for the considered pair of distinct SF code matrices, we have

$$\begin{aligned}&\operatorname{rank}\left(\mathbf{U}^b(\mathbf{X} - \hat{\mathbf{X}})\mathbf{\Lambda}(\mathbf{X} - \hat{\mathbf{X}})^{\mathcal{H}}\mathbf{U}^{b\mathcal{H}}\right)\\ \le\ &\operatorname{rank}\left(\mathbf{U}^b(\mathbf{X} - \hat{\mathbf{X}})\right)\\ =\ &\operatorname{rank}\left(\mathbf{F}_N\bar{\mathbf{V}}_1\bar{\mathbf{P}}^b_1\Delta\mathbf{X}_1\right)\\ \le\ &\operatorname{rank}\left(\bar{\mathbf{V}}_1\right)\\ <\ &M_t\Gamma.\end{aligned} \tag{26}$$

This implies that diversity order $M_t\Gamma$ cannot be achieved.

D. Diversity Analysis with the ZF-ML Detection Method

In the above analysis, by treating the ICI terms due to CFOs as a part of an SF code matrix, we find that the achieved diversity order of the SF codes is not decreased by CFOs under the condition that the absolute values of normalized CFOs are less than 0.5. This property is based on the ML decoding where it is required to jointly consider all the N subcarriers when decoding the the new (virtual) code after absorbing the CFOs/ICI into the original SF code. Given a not so small N (true in practice), the ML decoding complexity will be high even for efficient ML detection methods such as the sphere decoding method. We next consider the SF coded system after we equalize the ICI caused by the CFOs using the ZF method, i.e., the two-stage ZF aided ML (ZF-ML) method. The ZF-ML method is described as follows.

Let us recall the signal model (15). As we have analyzed that if $\varepsilon_{Max} < 0.5$, the ICI matrix $\mathbf{U}^b$ is nonsingular. Under this condition we can equalize the ICI matrix $\mathbf{U}^b$ by the ZF method. Thus, after multiplying $\mathbf{z}^b$ by $\mathbf{U}^{b^{-1}}$, we obtain the ICI free signal model

$$\tilde{\mathbf{z}}^b = \mathbf{U}^{b^{-1}}\mathbf{z}^b = \sqrt{\frac{\rho}{M_t}}\mathbf{X}\mathbf{h} + \tilde{\mathbf{w}}, \tag{27}$$

where $\tilde{\mathbf{w}} = \mathbf{U}^{b^{-1}}\mathbf{w}$. Then define the $(p-1)$th, $1 \le p \le P$, $M_t\Gamma \times 1$ subvector of $\tilde{\mathbf{z}}^b$ as $\tilde{\mathbf{z}}^b_p$ and each $\tilde{\mathbf{z}}^b_p$ is given by

$$\tilde{\mathbf{z}}^b_p = \sqrt{\frac{\rho}{M_t}}\mathbf{X}_p\mathbf{h} + \tilde{\mathbf{w}}_p, \tag{28}$$

where $\mathbf{X}_p$ is defined in Section IV-C, $\tilde{\mathbf{w}}_p = \hat{\mathbf{U}}^b_p\mathbf{w}$ is the $(p-1)$th $M_t\Gamma \times 1$ subvector of $\tilde{\mathbf{w}}$ and $\hat{\mathbf{U}}^b_p$ is the $(p-1)$th $M_t\Gamma \times N$ submatrix of $\mathbf{U}^{b^{-1}}$. Thus the covariance matrix of the new noise $\tilde{\mathbf{w}}_p$ is given by $\mathbf{T}_{b,p} = \mathrm{E}\left[\tilde{\mathbf{w}}_p\tilde{\mathbf{w}}^{\mathcal{H}}_p\right] = \hat{\mathbf{U}}^b_p\left(\hat{\mathbf{U}}^b_p\right)^{\mathcal{H}}$. As $\hat{\mathbf{U}}^b_p$ has full row rank $M_t\Gamma$, it is clear that $\mathbf{T}_{b,p}$ is nonsingular. So for the signal model (28), if we decode the SF submatrix $\mathbf{G}_p$ by the ML criterion, the achieved diversity order is still equal to the minimum rank of the matrix $(\mathbf{X}_p - \hat{\mathbf{X}}_p)\mathbf{\Lambda}(\mathbf{X}_p - \hat{\mathbf{X}}_p)^{\mathcal{H}}$ for any distinct $\mathbf{G}_p$ and $\hat{\mathbf{G}}_p$. Now, we are in a position to state the following theorem.

Theorem 2: For the SF code described in Section III, if the absolute values of normalized CFOs ε_m, $1 \le m \le M_t$, are all less than 0.5, then the ZF-ML detection method can still achieve the same diversity order as the case without CFOs.

Proof: It directly follows from the above argument. ■

V. Full Diversity by Permutations

We have seen that if $\varepsilon_{Max} < 0.5$, the diversity order of the SF codes $\mathbf{C}$ is the same as the case without CFOs. On the other hand, if $\varepsilon_{Max} \ge 0.5$, it is possible that the ICI matrix $\mathbf{U}^b$ is singular and then full diversity may not be achieved. Note that in (16) the singularity of $\mathbf{U}^b$ may not necessarily lead to the inequality if the rank of $\mathbf{U}^b$ is not smaller than the rank of $(\mathbf{X} - \hat{\mathbf{X}})\mathbf{\Lambda}(\mathbf{X} - \hat{\mathbf{X}})^{\mathcal{H}}$. Hence during the analysis of the counterexample for Theorem 1 in Section IV-C, we need to show that in (22) a submatrix of $\mathbf{V}$, i.e, $\bar{\mathbf{V}}_1$, has no full column rank $M_t\Gamma$ when $\mathbf{V}$ is singular. On the other hand, from the expression of $\mathbf{V}$ in (17), we know that the matrix $\mathbf{V}$ depends on the *selection* matrix $\mathbf{P}_m$ for $1 \le m \le M_t$, which can be changed by permuting the SF code matrix $\mathbf{C}$. This motivates us that the full diversity property of the SF codes may be enhanced by a permutation (interleaving) method. We explore this possibility in this section and propose a permutation method to guarantee full diversity even when $\mathbf{U}^b$ is singular. Throughout this section, the superscript $(\cdot)'$ of a matrix or vector means that the matrix or vector is defined for the permuted SF code $\mathbf{C}'$ as a counterpart of it defined for $\mathbf{C}$ in Section IV.

$$\mathbf{V} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ e^{\frac{j2\pi\varepsilon_1}{8}} & e^{\frac{j2\pi(\varepsilon_1+1)}{8}} & e^{\frac{j2\pi(\varepsilon_2+2)}{8}} & e^{\frac{j2\pi(\varepsilon_2+3)}{8}} & e^{\frac{j2\pi(\varepsilon_1+4)}{8}} & e^{\frac{j2\pi(\varepsilon_1+5)}{8}} & e^{\frac{j2\pi(\varepsilon_2+6)}{8}} & e^{\frac{j2\pi(\varepsilon_2+7)}{8}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ e^{\frac{j2\pi\varepsilon_1 7}{8}} & e^{\frac{j2\pi(\varepsilon_1+1)7}{8}} & e^{\frac{j2\pi(\varepsilon_2+2)7}{8}} & e^{\frac{j2\pi(\varepsilon_2+3)7}{8}} & e^{\frac{j2\pi(\varepsilon_1+4)7}{8}} & e^{\frac{j2\pi(\varepsilon_1+5)7}{8}} & e^{\frac{j2\pi(\varepsilon_2+6)7}{8}} & e^{\frac{j2\pi(\varepsilon_2+7)7}{8}} \end{bmatrix} \quad (24)$$

$$\mathbf{V}' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ e^{\frac{j2\pi\varepsilon_1}{8}} & e^{\frac{j2\pi(\varepsilon_1+1)}{8}} & e^{\frac{j2\pi(\varepsilon_1+2)}{8}} & e^{\frac{j2\pi(\varepsilon_1+3)}{8}} & e^{\frac{j2\pi(\varepsilon_2+4)}{8}} & e^{\frac{j2\pi(\varepsilon_2+5)}{8}} & e^{\frac{j2\pi(\varepsilon_2+6)}{8}} & e^{\frac{j2\pi(\varepsilon_2+7)}{8}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ e^{\frac{j2\pi\varepsilon_1 7}{8}} & e^{\frac{j2\pi(\varepsilon_1+1)7}{8}} & e^{\frac{j2\pi(\varepsilon_1+2)7}{8}} & e^{\frac{j2\pi(\varepsilon_1+3)7}{8}} & e^{\frac{j2\pi(\varepsilon_2+4)7}{8}} & e^{\frac{j2\pi(\varepsilon_2+5)7}{8}} & e^{\frac{j2\pi(\varepsilon_2+6)7}{8}} & e^{\frac{j2\pi(\varepsilon_2+7)7}{8}} \end{bmatrix} \quad (25)$$

A. Proposed Permutation Method

At the transmitter, we re-arrange the SF code $\mathbf{C}$ constructed by (2)-(3) by some row permutations and denote this row-wisely permuted SF code by $\mathbf{C}'$, which is given by

$$\begin{aligned} \mathbf{C}' = \; & \sqrt{M_t}\mathrm{diag}([\mathbf{x}_1^{1\mathcal{T}}, \mathbf{x}_1^{2\mathcal{T}}, \cdots, \mathbf{x}_1^{P\mathcal{T}}]^{\mathcal{T}}, \\ & [\mathbf{x}_2^{1\mathcal{T}}, \mathbf{x}_2^{2\mathcal{T}}, \cdots, \mathbf{x}_2^{P\mathcal{T}}]^{\mathcal{T}}, \\ & \cdots, [\mathbf{x}_{M_t}^{1\;\mathcal{T}}, \mathbf{x}_{M_t}^{2\;\mathcal{T}}, \cdots, \mathbf{x}_{M_t}^{P\;\mathcal{T}}]^{\mathcal{T}}), \end{aligned} \quad (29)$$

where $\mathbf{x}_m^p$ is defined in (3). From (29), we can see that after row permutations, the nonzero entries of the $(m-1)$th, $1 \le m \le M_t$, column of $\mathbf{C}$ are grouped and the subcarriers from $P\Gamma(m-1)$ to $P\Gamma m - 1$ are used by the mth transmit antenna. We call these subcarriers the mth group. Similar to the definition of matrix $\mathbf{P}_m$ defined in (5), for $\mathbf{C}'$ we also define the diagonal *selection* matrix $\mathbf{P}'_m$ for $1 \le m \le M_t$, corresponding to the $(m-1)$th column of $\mathbf{C}'$ with the form

$$\mathbf{P}'_m(l, l') = \begin{cases} 1, & \text{if } l = l' \text{ and } (m-1)P\Gamma \le l \le mP\Gamma - 1 \\ 0, & \text{else} \end{cases}, \quad (30)$$

for $0 \le l, l' \le N-1$. Different from $\mathbf{P}_m$ in (5), due to the permutation, the nonzero diagonal entries of $\mathbf{P}'_m$ are grouped as well as the nonzero entries of $(m-1)$th column of $\mathbf{C}'$.

In [11], it has been shown that the diversity order and coding advantage of this family of SF codes only depend on the relative positions of the permuted rows corresponding to the entries of $\mathbf{x}_m^p$ in (3) with respect to the position of $x^p_{(m-1)\Gamma+1}$. As we do not change the relative positions for the entries of $\mathbf{x}_m^p$ for each pair of m and p, $\mathbf{C}'$ should have the same diversity order and coding advantage as those of $\mathbf{C}$.

B. Receive Signal Model with Permutation

When applying $\mathbf{C}'$ to the cooperative protocol described in Section II-A, analogous to the derivation of (15), we can get the following SF coded receive signal model:

$$\mathbf{z}'^b = \sqrt{\frac{\rho}{M_t}}\mathbf{U}'^b\mathbf{X}'\mathbf{h} + \mathbf{w}, \quad (31)$$

where $\mathbf{X}' = [\mathbf{J}_0 \odot \mathbf{C}', \cdots, \mathbf{J}_{L-1} \odot \mathbf{C}']$, $\mathbf{U}'^b = \sum_{m=1}^{M_t} e^{j\theta^b_{\varepsilon_m}}\mathbf{U}_{\varepsilon_m}\mathbf{P}'_m$. Then $\mathbf{U}'^b$ can be further rewritten as

$$\mathbf{U}'^b = N^{-\frac{1}{2}}\mathbf{F}_N\mathbf{V}'\mathbf{P}'^b, \quad (32)$$

where $\mathbf{V}' = N^{\frac{1}{2}} \sum_{m=1}^{M_t} \mathbf{\Omega}_{\varepsilon_m}\mathbf{F}_N^{\mathcal{H}}\mathbf{P}'_m$ and the diagonal unitary matrix $\mathbf{P}'^b$ is given by $\mathbf{P}'^b = \sum_{m=1}^{M_t} e^{j\theta^b_{\varepsilon_m}}\mathbf{P}'_m$. Considering the expressions of $\mathbf{\Omega}_{\varepsilon_m}$ in (8) and $\mathbf{P}'_m$ defined in (30), we can derive that the kth, $0 \le k \le N-1$, column of $\mathbf{V}'$ (denoted by $\mathbf{v}'_k$) is the kth column of $N^{\frac{1}{2}}\mathbf{\Omega}_{\varepsilon_{m_k}}\mathbf{F}_N^{\mathcal{H}}$ and has the same expression as that of $\mathbf{v}_k$ by

$$\mathbf{v}'_k = \mathbf{q}^k_{\varepsilon_{m_k}} = [1, e^{j2\pi(\varepsilon_{m_k}+k)/N}, \cdots, e^{j2\pi(\varepsilon_{m_k}+k)(N-1)/N}]^{\mathcal{T}}. \quad (33)$$

Here m_k still denotes the kth subcarrier used by the m_kth transmit antenna. For $\mathbf{C}'$, m_k is different from that of $\mathbf{C}$ and is given by

$$m_k = \lceil (k+1)/(P\Gamma) \rceil \text{ for } 0 \le k \le N-1. \quad (34)$$

Following the way from (19) to (20), we can see that $\mathbf{V}'$ is also a Vandermonde matrix. Then the sufficient and necessary condition for $\mathbf{V}'$ to be singular is given by

$$\det(\mathbf{V}') = 0 \Leftrightarrow \exists\ i, l, k \text{ that } \varepsilon_{m_l} - \varepsilon_{m_i} = tN + i - l \Leftrightarrow \mathbf{v}'_i = \mathbf{v}'_l, \quad (35)$$

for $0 \le i < l \le N-1$ and t, l and i are all integers.

C. Motivation of Proposed Permutation Method

Due to the proposed permutation method, the *selection* matrix $\mathbf{P}'_m$ in (30) is different from $\mathbf{P}_m$ in (5) defined for $\mathbf{C}$, which leads to the difference between $\mathbf{V}$ and $\mathbf{V}'$. From (5), (17) and (19), $\mathbf{V}$ has the form as (36) at the top of next page. Clearly, all the columns of $\mathbf{V}$ are divided into PM_t groups and the lth, $1 \le l \le PM_t$, group is just the $(l-1)$th $N \times \Gamma$ submatrix of $N^{\frac{1}{2}}\mathbf{\Omega}_{\varepsilon_{(l-1)_{M_t}+1}}\mathbf{F}_N^{\mathcal{H}}$. On the other hand, from (30), (32) and (33), $\mathbf{V}'$ has the form as (37) where due to the permutation, all of its N columns are divided into M_t *big* groups and the lth, $1 \le l \le M_t$, group is just the $(l-1)$th $N \times P\Gamma$ submatrix of $N^{\frac{1}{2}}\mathbf{\Omega}_{\varepsilon_l}\mathbf{F}_N^{\mathcal{H}}$.

Similar to the definition of $\mathbf{X}_p$, $\bar{\mathbf{V}}_p$ and $\bar{\mathbf{P}}^b_p$ in (22), for the permuted SF code $\mathbf{C}'$, we also define $\mathbf{X}'_p$, $\bar{\mathbf{V}}'_p$ and $\bar{\mathbf{P}}'^b_p$ for $1 \le p \le P$. $\mathbf{X}'_p$ is an $M_t\Gamma \times M_tL$ submatrix of $\mathbf{X}'$ in (31) and contains all the rows of $\mathbf{X}'$ related to $\mathbf{G}_p$ given by (3). $\bar{\mathbf{V}}'_p$ contains $M_t\Gamma$ columns of $\mathbf{V}'$ whose column indexes are equal to the row indexes of the rows of $\mathbf{X}'$ contained by $\mathbf{X}'_p$. From (29) and the expression of $\mathbf{X}'$, it is easy to see that the $(m-1)$th $N \times \Gamma$ submatrix of $\bar{\mathbf{V}}'_p$ is just the $(p-1)$th $N \times \Gamma$ submatrix of the mth group of $\mathbf{V}'$ in (37). For example, $\bar{\mathbf{V}}'_1$ has the form as (38) at the top of next page. $\bar{\mathbf{P}}^b_p$ is an $M_t\Gamma \times M_t\Gamma$ diagonal matrix which

$$\mathbf{V} = [\underbrace{\mathbf{q}_{\varepsilon_1}^{0}, \cdots, \mathbf{q}_{\varepsilon_1}^{\Gamma-1}}_{\text{1st } \Gamma \text{ columns}}, \underbrace{\mathbf{q}_{\varepsilon_2}^{\Gamma}, \cdots, \mathbf{q}_{\varepsilon_2}^{2\Gamma-1}}_{\text{2nd } \Gamma \text{ columns}}, \cdots, \underbrace{\mathbf{q}_{\varepsilon_{M_t}}^{(M_t-1)\Gamma}, \cdots, \mathbf{q}_{\varepsilon_{M_t}}^{M_t\Gamma-1}}_{M_t\text{th } \Gamma \text{ columns}}, \underbrace{\mathbf{q}_{\varepsilon_1}^{M_t\Gamma}, \cdots, \mathbf{q}_{\varepsilon_1}^{(M_t+1)\Gamma-1}}_{(M_t+1)\text{th } \Gamma \text{ columns}}, \cdots, \underbrace{\mathbf{q}_{\varepsilon_{M_t}}^{(PM_t-1)\Gamma}, \cdots, \mathbf{q}_{\varepsilon_{M_t}}^{PM_t\Gamma-1}}_{PM_t\text{th } \Gamma \text{ columns}}] \quad (36)$$

$$\mathbf{V}' = [\underbrace{\mathbf{q}_{\varepsilon_1}^{0}, \mathbf{q}_{\varepsilon_1}^{1}, \cdots, \mathbf{q}_{\varepsilon_1}^{P\Gamma-1}}_{\text{1st } P\Gamma \text{ columns}}, \cdots, \underbrace{\mathbf{q}_{\varepsilon_{m^\dagger}}^{(m^\dagger-1)P\Gamma}, \cdots, \mathbf{q}_{\varepsilon_{m^\dagger}}^{m^\dagger P\Gamma-1}}_{m^\dagger\text{th } P\Gamma \text{ columns}}, \underbrace{\mathbf{q}_{\varepsilon_{(m^\dagger+1)}}^{m^\dagger P\Gamma}, \cdots, \mathbf{q}_{\varepsilon_{(m^\dagger+1)}}^{(m^\dagger+1)P\Gamma-1}}_{(m^\dagger+1)\text{th } P\Gamma \text{ columns}}, \cdots, \underbrace{\mathbf{q}_{\varepsilon_{M_t}}^{(M_t-1)P\Gamma}, \cdots, \mathbf{q}_{\varepsilon_{M_t}}^{M_tP\Gamma-1}}_{M_t\text{th } P\Gamma \text{ columns}}] \quad (37)$$

$$\bar{\mathbf{V}}_1' = [\underbrace{\mathbf{q}_{\varepsilon_1}^{0}, \cdots, \mathbf{q}_{\varepsilon_1}^{\Gamma-1}}_{\text{1st } \Gamma \text{ columns}}, \cdots, \underbrace{\mathbf{q}_{\varepsilon_{m^\dagger}}^{(m^\dagger-1)P\Gamma}, \cdots, \mathbf{q}_{\varepsilon_{m^\dagger}}^{(m^\dagger-1)P\Gamma+\Gamma-1}}_{m^\dagger\text{th } \Gamma \text{ columns}}, \underbrace{\mathbf{q}_{\varepsilon_{m^\dagger+1}}^{m^\dagger P\Gamma}, \cdots, \mathbf{q}_{\varepsilon_{m^\dagger+1}}^{m^\dagger P\Gamma+\Gamma-1}}_{(m^\dagger+1)\text{th } \Gamma \text{ columns}}, \cdots, \underbrace{\mathbf{q}_{\varepsilon_{M_t}}^{(M_t-1)P\Gamma}, \cdots, \mathbf{q}_{\varepsilon_{M_t}}^{(M_t-1)P\Gamma+\Gamma-1}}_{M_t\text{th } \Gamma \text{ columns}}] \quad (38)$$

contains $M_t\Gamma$ diagonal entries of $\mathbf{P}^{b'}$ corresponding to the rows of $\mathbf{X}'$ contained by $\mathbf{X}_p'$. As a consequence, we have

$$\mathbf{V}'\mathbf{P}^{b'}\mathbf{X}' = \sum_{p=1}^{P} \bar{\mathbf{V}}_p'\bar{\mathbf{P}}_p'^{b}\mathbf{X}_p'. \quad (39)$$

Let us recall the counter example illustrated in Section IV-C. For $\mathbf{C}'$, we also consider the case that $\mathbf{G}_p = \hat{\mathbf{G}}_p$ for $2 \le p \le P$ and $\varepsilon_{m^\dagger+1} - \varepsilon_{m^\dagger} = -1$. In this case the achieved diversity order of $\mathbf{C}'$ is upper bounded by the rank of $\bar{\mathbf{V}}_1'$. Therefore, if $\bar{\mathbf{V}}_1'$ does not have full column rank $M_t\Gamma$, full diversity cannot be achieved. However, different from $\bar{\mathbf{V}}_1$ in this case, although $\mathbf{V}'$ is singular, $\bar{\mathbf{V}}_1'$ still has full column rank $M_t\Gamma$. By checking (35), we can see that when $\varepsilon_{m^\dagger+1} - \varepsilon_{m^\dagger} = -1$, in (37) the first column of $(m^\dagger + 1)$th group of $\mathbf{V}'$, i.e., $\mathbf{q}_{\varepsilon_{m^\dagger+1}}^{m^\dagger P\Gamma}$ is equal to the last column of $m^\dagger$th group of $\mathbf{V}'$, i.e., $\mathbf{q}_{\varepsilon_{m^\dagger}}^{m^\dagger P\Gamma-1}$. Hence $\mathbf{V}'$ is singular. However, due to the permutation, $\bar{\mathbf{V}}_1'$ does not contain the column $\mathbf{q}_{\varepsilon_{m^\dagger}}^{m^\dagger P\Gamma-1}$ as shown in (38). Therefore, $\bar{\mathbf{V}}_1'$ still has full column rank. Corresponding to (24), we also give the structure of $\mathbf{V}'$ for $M_t = 2$, $N = 8$ and $\Gamma = 2$ as (25) at the top of previous page. By substituting $\varepsilon_2 = \varepsilon_1 - 1$ into (25), we can clearly see that although $\mathbf{V}'$ is singular, both $\bar{\mathbf{V}}_1'$, which contains 0th, 1st, 4th, and 5th columns of $\mathbf{V}'$, and $\bar{\mathbf{V}}_2'$, which contains 2nd 3rd, 6th, and 7th columns of $\mathbf{V}'$, still have full column rank 4. Therefore the counter example illustrated in Section IV-C does not hold for $\mathbf{C}'$.

The above discussions only show us the possibility that the permuted SF codes $\mathbf{C}'$ may still achieve full diversity even when the ICI matrix is singular, since $\text{rank}(\bar{\mathbf{V}}_p') = M_t\Gamma$ for $1 \le p \le P$ is not sufficient to derive $\text{rank}(\sum_{p=1}^{P} \bar{\mathbf{V}}_p'\bar{\mathbf{P}}_p'^{b}\mathbf{X}_p') = M_t\Gamma$. Fortunately, by utilizing the properties of $\mathbf{V}'$ and $\mathbf{C}'$, we find a sufficient condition under which the permuted SF codes can obtain full diversity even when the ICI matrix $\mathbf{U}'^{b}$ is singular.

D. Properties of $\mathbf{V}'$ *When* $|\varepsilon_m| < \frac{(P-1)\Gamma+1}{2}$

Before illustrating our major result, let us first see some properties of $\mathbf{V}'$ when $|\varepsilon_m| < \frac{(P-1)\Gamma+1}{2}$ and $P \ge 2$, which will be utilized in the remainder of this section. Here, we define some additional notations. Define the $(m-1)$th, $1 \le m \le M_t$, $N \times P\Gamma$ submatrix of $\mathbf{V}'$ as $\mathbf{V}_m'$, i.e., $\mathbf{V}' = [\mathbf{V}_1', \mathbf{V}_2', \cdots, \mathbf{V}_{M_t}']$. The $(p-1)$th, $1 \le p \le P$, $N \times \Gamma$ submatrix of $\mathbf{V}_m'$ is further denoted by $\mathbf{V}'^{p}_{m}$, i.e., $\mathbf{V}_m' = [\mathbf{V}'^{1}_{m}, \mathbf{V}'^{2}_{m}, \cdots, \mathbf{V}'^{P}_{m}]$.

To derive the properties of $\mathbf{V}'$, we need to utilize the following proposition, the proof of which can be found in Appendix A.

Proposition 1: For the matrix $\mathbf{V}'$ defined in (32), provided $|\varepsilon_m| < \frac{(P-1)\Gamma+1}{2}$ for $1 \le m \le M_t$ and $P > 1$, $\mathbf{V}'$ is singular if and only if there exist at least a pair of integers i and l for $0 \le i < l \le N-1$ and $m_i \ne m_l$ that satisfy either one of the following two cases:

- Case 1: $\varepsilon_{m_l} - \varepsilon_{m_i} = i - l$ for $\varepsilon_{m_l} - \varepsilon_{m_i} \in \{-(P-1)\Gamma, -(P-1)\Gamma + 1, \cdots, -1\}$ and $m_l - m_i = 1$.
- Case 2: $\varepsilon_{m_l} - \varepsilon_{m_i} = N + i - l$ for $\varepsilon_{m_l} - \varepsilon_{m_i} \in \{1, 2, \cdots, (P-1)\Gamma\}$, $m_i = 1$ and $m_l = M_t$.

In both of these two cases, $\mathbf{v}_i' = \mathbf{v}_l'$. Here, ε_m is the normalized CFO, m_k denotes kth, $0 \le k \le N-1$, subcarrier is used by m_kth transmit antenna and $\mathbf{v}_k'$ is the kth column of $\mathbf{V}'$.

As all the columns of $\mathbf{V}_m$, $1 \le m \le M_t$, are related to ε_m, according to Proposition 1, some columns of $\mathbf{V}_m$ may be repeated in $\mathbf{V}_{(m)_{M_t}+1}'$ or $\mathbf{V}_{(m-2)_{M_t}+1}'$. So we can define a set $\mathcal{B}_m$ for $\mathbf{V}_m'$ as

$$\mathcal{B}_m = \{t_{m,1}, t_{m,2}, \cdots, t_{m,T_m}\} \text{ for } t_{m,1} < t_{m,2} < \cdots < t_{m,T_m}, \quad (40)$$

where $T_m = |\mathcal{B}_m|$, so that $\mathcal{B}_m$ contains all the indices of the columns of $\mathbf{V}'$ which belong to $\mathbf{V}_m'$ and are equal to some columns of $\mathbf{V}_{(m)_{M_t}+1}'$. Thus, if $\mathcal{B}_m$ is not null, the columns $\mathbf{v}_{t_{m,l}}'$ for $1 \le l \le T_m$, which are contained by $\mathbf{V}_m'$, are repeated in $\mathbf{V}_{(m)_{M_t}+1}'$. Similarly, we also define a set $\bar{\mathcal{B}}_m$ containing all of the indices of the columns of $\mathbf{V}'$ which belong to $\mathbf{V}_m'$ and are repeated in $\mathbf{V}_{(m-2)_{M_t}+1}'$. $\bar{\mathcal{B}}_m$ is given by

$$\bar{\mathcal{B}}_m = \{\bar{t}_{m,1}, \bar{t}_{m,2}, \cdots, \bar{t}_{m,\bar{T}_m}\} \text{ for } \bar{t}_{m,1} < \bar{t}_{m,2} < \cdots < \bar{t}_{m,\bar{T}_m}, \quad (41)$$

where $\bar{T}_m = |\bar{\mathcal{B}}_m|$. We further define a function $f_m(t)$ for $t \in \mathcal{B}_m$, which is used to calculate the index of the column of $\mathbf{V}'$ contained by $\mathbf{V}_{(m)_{M_t}+1}'$ so that $\mathbf{v}_t' = \mathbf{v}_{f_m(t)}'$. According to Proposition 1, $f_m(t)$ has the following form

$$f_m(t) = \begin{cases} t - \varepsilon_{m+1} + \varepsilon_m & \text{for } 1 \le m \le M_t - 1 \\ t - N + \varepsilon_{M_t} - \varepsilon_1 & \text{for } m = M_t \end{cases}. \quad (42)$$

Thus $\bar{\mathcal{B}}_m$ is the image of $\mathcal{B}_{(m-2)_{M_t}+1}$ under the injective function $f_{(m-2)_{M_t}+1}$, when $\mathcal{B}_{(m-2)_{M_t}+1}$ is not null.

As an example, Fig. 2 shows a part of the structure of $\mathbf{V}'$ when $\mathcal{B}_m$ is not null, where each pair of equal columns are connected by a line. To gain an insight into $\mathbf{V}'$, from

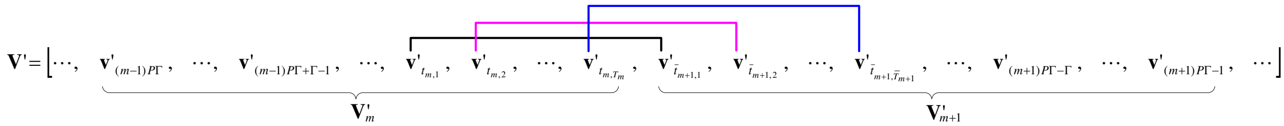

Fig. 2. Structure of $\mathbf{V}'$ when $\mathcal{B}_{m^\dagger}$ is not null.

Proposition 1, we can obtain the following *Properties* of $\mathbf{V}'$ in (32), when $P \geq 1$ and $\mathcal{B}_m$ or $\bar{\mathcal{B}}_m$ is not null.

1) $mP\Gamma - (P-1)\Gamma \leq t_{m,1}$
2) $f_m(t_{m,1}) = (m)_{M_t}P\Gamma$, which means that $\mathbf{v}'_{t_{m,1}}$ is equal to the first column of $\mathbf{V}'_{(m)_{M_t}}$.
3) $\mathcal{B}_m$ has the form $\mathcal{B}_m = \{t_{m,1}, t_{m,1}+1, \cdots, mP\Gamma - 1\}$.
4) $\bar{t}_{m,\bar{T}_m} \leq mP\Gamma - \Gamma - 1$.
5) $\bar{\mathcal{B}}_m$ has the form

$$\begin{aligned}\bar{\mathcal{B}}_m &= \{(m-1)P\Gamma, (m-1)P\Gamma+1, \cdots, \\ &\quad (m-1)P\Gamma + \bar{T}_m - 1\} \\ &= \{f_{m^\dagger}(t_{m^\dagger,1}), f_{m^\dagger}(t_{m^\dagger,2}), \cdots, f_{m^\dagger}(t_{m^\dagger,T_{m^\dagger}})\},\end{aligned} \tag{43}$$

where $m^\dagger = (m-2)_{M_t} + 1$.

6) If both $\bar{\mathcal{B}}_m$ and $\mathcal{B}_m$ are not null sets, we must have $t_{m,1} - \bar{t}_{m,\bar{T}_m} \geq \Gamma + 1$.

From *Property* 6 of $\mathbf{V}'$, we know that the intersection of $\bar{\mathcal{B}}_m$ and $\mathcal{B}_m$ is always a null set, which implies that in $\mathbf{V}'$ we cannot find more than two equal columns. Otherwise, there must exist m so that $\bar{\mathcal{B}}_m \cap \mathcal{B}_m \neq \phi$, which is contradictory to *Property* 6 of $\mathbf{V}'$.
7) There must exist $m^\dagger$ so that $\mathcal{B}_{m^\dagger} = \phi$.
8) There must exist $m^\dagger$ so that $\bar{\mathcal{B}}_{m^\dagger} = \phi$

The proofs of these properties can be found in Appendix A.

E. Diversity Analysis of Permuted SF Codes

The sufficient condition, under which multiple CFOs do not affect the diversity order of the permuted SF Codes $\mathbf{C}'$, is given by the following theorem:

Theorem 3: Let ε_m be the normalized CFO. If $|\varepsilon_m|$ are all less than $\frac{(P-1)\Gamma+1}{2}$ for $1 \leq m \leq M_t$ and P is larger than 1 , the permuted SF code described in (29) can achieve the same diversity order as the case without CFOs.

Property of SF Codes: To prove Theorem 3, besides the properties of $\mathbf{V}'$ derived in Section V-D, we need to utilize one property of the SF code as shown below.

From (13), we know that the $(m-1)$th, $1 \leq m \leq M_t$, column of $\mathbf{J}_l \odot \mathbf{C}'$ for $0 \leq l \leq L-1$ is $\mathbf{f}^{\tau_m(l)} \odot \mathbf{c}'_m$, where $\mathbf{c}'_m$ is the $(m-1)$th column of $\mathbf{C}'$ in (29). Define $\mathbf{y}^p_{m,l} = \sqrt{M_t}\left(\mathbf{f}^{\tau_m(l)}_{(m-1)P+p-1} \odot \mathbf{x}^p_m\right)$, where $\mathbf{f}^{\tau_m(l)}_k$, $0 \leq k \leq M_tP-1$, denotes kth $\Gamma \times 1$ subvector of $\mathbf{f}^{\tau_m(l)}$. Then we can express $\mathbf{J}_l \odot \mathbf{C}'$ as (44) at the top of next page. Observing (44), we can see that for the $(m-1)$th, $1 \leq m \leq M_t$, $P\Gamma \times M_t$ submatrix of $\mathbf{J}_l \odot \mathbf{C}'$, only its $(m-1)$th column, i.e., $[\mathbf{y}^{1\ \mathcal{T}}_{m,l}, \cdots, \mathbf{y}^{P\ \mathcal{T}}_{m,l}]^{\mathcal{T}}$, has nonzero entries. For each m, by collecting all these nonzero $P\Gamma \times 1$ columns from $\mathbf{J}_l \odot \mathbf{C}'$ for $0 \leq l \leq L-1$, we define the $P\Gamma \times L$ matrix $\mathbf{Y}_m$, $1 \leq m \leq M_t$, by

$$\mathbf{Y}_m = \begin{bmatrix} \mathbf{y}^1_{m,0} & \mathbf{y}^1_{m,1} & \cdots & \mathbf{y}^1_{m,L-1} \\ \mathbf{y}^2_{m,0} & \mathbf{y}^2_{m,1} & \cdots & \mathbf{y}^2_{m,L-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{y}^P_{m,0} & \mathbf{y}^P_{m,1} & \cdots & \mathbf{y}^P_{m,L-1} \end{bmatrix}, \tag{45}$$

and denote the $(p-1)$th, $1 \leq p \leq P$, $\Gamma \times L$ submatrix of $\mathbf{Y}_m$ by $\mathbf{Y}^p_m$. Further define $\bar{\mathbf{Y}}_m$ and $\bar{\mathbf{Y}}^p_m$ as submatrices of $\mathbf{Y}_m$ and $\mathbf{Y}^p_m$, which contain the first $\min(\Gamma, L_m)$ columns of $\mathbf{Y}_m$ and $\mathbf{Y}^p_m$, respectively. Let $\Delta\bar{\mathbf{Y}}_m$ be the difference matrix between $\bar{\mathbf{Y}}_m$ and $\hat{\bar{\mathbf{Y}}}_m$ which are constructed from $\mathbf{C}'$ and $\hat{\mathbf{C}}'$ according to (45), respectively. Then one property of this SF code [11] can be stated in the following proposition.

Proposition 2: If $\mathbf{G}_{p_0} \neq \hat{\mathbf{G}}_{p_0}$ with $1 \leq p_0 \leq P$, the difference matrix $\Delta\bar{\mathbf{Y}}^{p_0}_m$ has full column rank $\min(\Gamma, L_m)$.

Proof: It directly comes from the full diversity property of this family of SF codes [2], [11]. ■

From the properties of $\mathbf{V}'$ and SF codes, we can obtain the following lemma:

Lemma 1: For integers n_1, n_2 with $0 \leq n_1 < n_2 \leq M_t$ and $\min(\Gamma, L_m) \times 1$ vectors $\mathbf{a}_m$, if either $n_2 \geq n_1 + 2$ or $\bar{\mathcal{B}}_{n_2} = \phi$ where $\bar{\mathcal{B}}_{n_2}$ is defined in (41), then $\sum_{m=1}^{n_1} \mathbf{V}'_m \Delta\bar{\mathbf{Y}}_m \mathbf{a}_m + \sum_{m=n_2}^{M_t} \mathbf{V}'_m \Delta\bar{\mathbf{Y}}_m \mathbf{a}_m = \mathbf{0}_N$, only if $\mathbf{a}_{n_2} = \mathbf{0}_{\min(\Gamma, L_{n_2})}$ where the difference matrices $\Delta\bar{\mathbf{Y}}_m$, $1 \leq m \leq M_t$, are obtained from two distinct SF code matrices according to (45) and the definition of $\bar{\mathbf{Y}}_m$. Here $\sum_{m=1}^{n_1} \mathbf{V}'_m \Delta\bar{\mathbf{Y}}_m \mathbf{a}_m + \sum_{m=n_2}^{M_t} \mathbf{V}'_m \Delta\bar{\mathbf{Y}}_m \mathbf{a}_m$ turns to $\sum_{m=n_2}^{M_t} \mathbf{V}'_m \Delta\bar{\mathbf{Y}}_m \mathbf{a}_m$ when $n_1 = 0$.

Proof: See Appendix B. ■

With the aid of Lemma 1, we can prove Theorem 3 as follows:

Proof of Theorem 3: From (16), we know that CFOs cannot increase diversity order of SF codes. Hence to prove Theorem 3, it is sufficient to show that there exist $\sum_{m=1}^{M_t} \min(\Gamma, L_m)$ linearly independent columns of $\mathbf{U}'^b \Delta\mathbf{X}' \mathbf{\Lambda}^{\frac{1}{2}}$ over all pairs of distinct SF code matrices $\mathbf{C}'$ and $\hat{\mathbf{C}}'$ given $\varepsilon_{Max} < \frac{(P-1)\Gamma+1}{2}$, where $\Delta\mathbf{X}' = \mathbf{X}' - \hat{\mathbf{X}}'$ in (31), $\varepsilon_{Max} = \max_{1 \leq m \leq M_t}(|\varepsilon_m|)$ and $\mathbf{\Lambda} = E[\mathbf{h}\mathbf{h}^{\mathcal{H}}]$. Obviously, if $\mathbf{U}'^b$ is nonsingular, Theorem 3 holds similarly to Theorem 1. Therefore, we only need to consider the case when $\mathbf{U}'^b$ is singular. Furthermore, because of $\mathbf{U}' = N^{-\frac{1}{2}} \mathbf{F}_N \mathbf{V}' \mathbf{P}'^b$ in (32), we have $\text{rank}(\mathbf{U}' \Delta\mathbf{X}' \mathbf{\Lambda}^{\frac{1}{2}}) = \text{rank}(\mathbf{V}' \mathbf{P}'^b \Delta\mathbf{X}' \mathbf{\Lambda}^{\frac{1}{2}})$, since the normalized FFT matrix $\mathbf{F}_N$ is nonsingular. Hence, it is equivalent to prove that $\mathbf{V}' \mathbf{P}'^b \Delta\mathbf{X}' \mathbf{\Lambda}^{\frac{1}{2}}$ has $\sum_{m=1}^{M_t} \min(\Gamma, L_m)$ linearly independent columns. Note as $\mathbf{P}'^b$ is a diagonal matrix with full rank N, if we regard $\mathbf{P}'^b \mathbf{X}'$ as $\mathbf{X}'$, we can still get the Proposition 2. To simplify the notation, we just set $\mathbf{P}'^b$ as an identity matrix.

Due to the definition of $\bar{\mathbf{Y}}_m$ and the fact that for $\mathbf{X}'$ in (31), each $\mathbf{J}_l \odot \mathbf{C}'$ given by (44) is a block diagonal

$$\mathbf{J}_l \odot \mathbf{C}' = \begin{bmatrix} \mathbf{y}_{1,l}^{1\ \mathcal{T}} & \cdots & \mathbf{y}_{1,l}^{P\ \mathcal{T}} & \mathbf{0}_\Gamma^\mathcal{T} & \cdots & \mathbf{0}_\Gamma^\mathcal{T} & \cdots & \mathbf{0}_\Gamma^\mathcal{T} & \cdots & \mathbf{0}_\Gamma^\mathcal{T} \\ \mathbf{0}_\Gamma^\mathcal{T} & \cdots & \mathbf{0}_\Gamma^\mathcal{T} & \mathbf{y}_{2,l}^{1\ \mathcal{T}} & \cdots & \mathbf{y}_{2,l}^{P\ \mathcal{T}} & \cdots & \mathbf{0}_\Gamma^\mathcal{T} & \cdots & \mathbf{0}_\Gamma^\mathcal{T} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0}_\Gamma^\mathcal{T} & \cdots & \mathbf{0}_\Gamma^\mathcal{T} & \mathbf{0}_\Gamma^\mathcal{T} & \cdots & \mathbf{0}_\Gamma^\mathcal{T} & \cdots & \mathbf{y}_{M_t,l}^{1\ \mathcal{T}} & \cdots & \mathbf{y}_{M_t,l}^{P\ \mathcal{T}} \end{bmatrix}^\mathcal{T} \quad (44)$$

matrix, $\mathbf{V}'_m$ multiplying the lth column of $\bar{\mathbf{Y}}_m$, $0 \le l \le \min(\Gamma, L_m)-1$, is just the (lM_t+m-1)th column of $\mathbf{V}'\mathbf{X}'$. From the definition of the channel vector $\mathbf{h}$ in Section IV-A, the (lM_t+m-1)th diagonal entry of $\mathbf{\Lambda}$ is nonzero for $0 \le l \le L_m-1$. Hence the (lM_t+m-1)th column of $\mathbf{V}'\mathbf{X}'\mathbf{\Lambda}^{\frac{1}{2}}$ is only a scaled version of the (lM_t+m-1)th column of $\mathbf{V}'\mathbf{X}'$. As a consequence, in order to prove Theorem 3, it is sufficient to show that the $\sum_{m=1}^{M_t} \min(\Gamma, L_m)$ columns of $[\mathbf{V}'_1\Delta\bar{\mathbf{Y}}_1, \mathbf{V}'_2\Delta\bar{\mathbf{Y}}_2, \cdots, \mathbf{V}'_{M_t}\Delta\bar{\mathbf{Y}}_{M_t}]$ are linearly independent. Let $\mathbf{a}_m$ be an $\min(\Gamma, L_m)\times 1$ vector. In order to achieve our goal, we only need to show that $\sum_{m=1}^{M_t} \mathbf{V}'_m\Delta\bar{\mathbf{Y}}_m\mathbf{a}_m = \mathbf{0}_N$, only if $\mathbf{a}_m = \mathbf{0}_{\min(\Gamma,L_m)}$ for $1 \le m \le M_t$.

From *Property* 8 of $\mathbf{V}'$, we know that there exists an integer $m^\dagger$ with $1 \le m^\dagger \le M_t$, so that $\bar{\mathcal{B}}_{m^\dagger} = \phi$. Without loss of generality, we assume $\bar{\mathcal{B}}_1 = \phi$. According to Lemma 1, we have $\sum_{m=1}^{M_t} \mathbf{V}'_m\Delta\bar{\mathbf{Y}}_m\mathbf{a}_m = \mathbf{0}_N$, only if $\mathbf{a}_1 = \mathbf{0}_{\min(\Gamma,L_1)}$. As $\sum_{m=1}^{M_t} \mathbf{V}'_m\Delta\bar{\mathbf{Y}}_m\mathbf{a}_m = \mathbf{0}_N$ holds, only if $\mathbf{a}_1 = \mathbf{0}_{\min(\Gamma,L_1)}$, we can substitute $\mathbf{a}_1 = \mathbf{0}_{\min(\Gamma,L_1)}$ into $\sum_{m=1}^{M_t} \mathbf{V}'_m\Delta\bar{\mathbf{Y}}_m\mathbf{a}_m = \mathbf{0}_N$ and then obtain $\sum_{m=2}^{M_t} \mathbf{V}'_m\Delta\bar{\mathbf{Y}}_m\mathbf{a}_m = \mathbf{0}_N$, which corresponds to the case of $n_1 = 0$ and $n_2 = 2$ in Lemma 1. By applying Lemma 1 once more, we show that $\sum_{m=2}^{M_t} \mathbf{V}'_m\Delta\bar{\mathbf{Y}}_m\mathbf{a}_m = \mathbf{0}_N$, only if $\mathbf{a}_2 = \mathbf{0}_{\min(\Gamma,L_2)}$.

Actually, for each m with $2 \le m \le M_t$, by substituting $\mathbf{a}_n = \mathbf{0}_{\min(\Gamma,L_n)}$ for $1 \le n \le m-1$ into $\sum_{m=1}^{M_t} \mathbf{V}'_m\Delta\bar{\mathbf{Y}}_m\mathbf{a}_m$ and then applying Lemma 1, we can obtain $\mathbf{a}_m = \mathbf{0}_{\min(\Gamma,L_m)}$. So by repeating this strategy for additional $M_t - 2$ times, we finally get that if $\mathbf{C}' \neq \hat{\mathbf{C}}'$ and $\varepsilon_{Max} < \frac{(P-1)\Gamma+1}{2}$, then $\sum_{m=1}^{M_t} \mathbf{V}'_m\Delta\bar{\mathbf{Y}}_m\mathbf{a}_m = \mathbf{0}_N$, only if $\mathbf{a}_m = \mathbf{0}_{\min(\Gamma,L_m)}$ for $1 \le m \le M_t$ in turn. This implies the desired result. ■

On the other hand, by following the similar strategy used in Section IV-C, it can be shown that if $\varepsilon_{max} \ge \frac{(P-1)\Gamma+1}{2}$, the maximum achieved diversity order of SF codes may be less than $M_t\Gamma$, although $L_m \ge \Gamma$ for $1 \le m \le M_t$.

F. Signal Detection when the ICI Matrix U′ is Singular

In Section V-E, we show that the permuted SF codes $\mathbf{C}'$ can achieve full diversity even when the ICI matrix $\mathbf{U}'$ is singular. This property is based on the joint consideration of all the N subcarriers. However, in this case the ZF-ML detection method cannot be directly applied to reduce the decoding complexity. To overcome this problem, we propose two suboptimal detection methods, namely ZF-ML-Zn and ZF-ML-PIC, both of which can achieve the same diversity order as the case without CFOs even when $\mathbf{U}'$ is singular.

1) ZF-ML-Zn Method:

From *Property* 3 and *Property* 5 of $\mathbf{V}'$, we know that if $\mathbf{V}'$ is singular, there must exist an integer $\tilde{m}$ with $1 \le \tilde{m} \le M_t$ so that $\mathbf{v}'_{\bar{t}_{\tilde{m},k}} = \mathbf{v}'_{t_{m^\dagger,k}}$ for $1 \le k \le \bar{T}_{m^\dagger}$, where $m^\dagger = (\tilde{m}-2)_{M_t} + 1$, $\bar{t}_{\tilde{m},k}$ and $t_{m^\dagger,k}$ are defined in (41) and (40), respectively. As $\mathbf{v}'_{\bar{t}_{\tilde{m},k}}$ and $\mathbf{v}'_{t_{m^\dagger,k}}$ are respectively the left and right boundary columns of $\mathbf{V}'_{\tilde{m}}$ and $\mathbf{V}'_{m^\dagger}$, to make the ZF-ML method still applicable, we may not use the last n subcarriers of each group at the cost of some efficiency loss. We denote this detection method as the ZF-ML-Zn method, where n is the number of unused subcarriers of each group. So, total M_tn subcarriers are not used. Fig. 3 illustrates the structure of $\mathbf{V}'$ for the ZF-ML-Zn method when $\bar{T}_{\tilde{m}} = n$, where the equal columns are connected by lines. In the figure, the last n marked squares in each group stands for the columns related to unused subcarriers. Therefore, if $\bar{T}_{\tilde{m}} \le n$, all the $N - M_tn$ active columns of $\mathbf{V}'$, i.e., the columns related to used subcarriers, are still linearly independent. Accordingly, the ZF-ML method can be directly used. For this method, we have the following theorem:

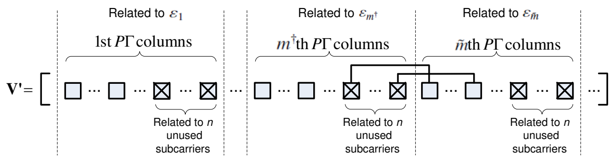

Fig. 3. Structure of $\mathbf{V}'$ for the ZF-ML-Zn method when $\bar{T}_{\tilde{m}} = n$

Theorem 4: Let ε_m be the normalized CFO. If $|\varepsilon_m|$, $1 \le m \le M_t$, are all less than $\frac{n+1}{2}$, for the permuted SF codes described in (29), the ZF-ML-Zn detection method can still achieve the same diversity order as the case without CFOs.

Proof: Similar to the derivation of *Property* 1 of $\mathbf{V}'$ under the condition $|\varepsilon_m| < \frac{(P-1)\Gamma+1}{2}$, provided $|\varepsilon_m| < \frac{n+1}{2}$, we can get that $mP\Gamma - n \le t_{m,1}$ for $1 \le m \le M_t$. From $t_{m,T_m} = mP\Gamma - 1$ (*Property* 3 of $\mathbf{V}'$), we can get $T_m = t_{m,T_m} - t_{m,1} + 1 \le n$ for $1 \le m \le M_t$. Therefore, just as the example illustrated in Fig. 3, the ZF-ML detection method can be directly used to achieve the same diversity order as the case without CFOs. ■

2) ZF-ML-PIC Detection Method:

To describe the ZF-ML-PIC method, by substituting (32) and (39) into (31), we rewrite $\mathbf{z}'^b$ as

$$\mathbf{z}'^b = \sqrt{\frac{\rho}{M_t}} N^{-\frac{1}{2}} \mathbf{F}_N \sum_{p=1}^{P} \bar{\mathbf{V}}'_p \bar{\mathbf{P}}'^b_p \mathbf{X}'_p \mathbf{h} + \mathbf{w}. \quad (46)$$

We then define a set $\bar{\mathcal{P}}$ which contains all the indices p with $1 \le p \le P$ so that the columns of $\bar{\mathbf{V}}'_p$ are unique in $\mathbf{V}'$, i.e., $\mathcal{B}_p = \bar{\mathcal{B}}_p = \phi$. When $\bar{\mathcal{P}}$ is not null, the ZF-ML-PIC method is described as follows:

1) Decode total $|\bar{\mathcal{P}}|$ SF submatrices $\mathbf{G}_p$ for $p \in \bar{\mathcal{P}}$ by the ZF-ML method.
Note that because $\mathbf{V}'$ is a Vandermonde matrix, all the distinct columns of $\mathbf{V}'$ are linearly independent. So it is not hard to see that we can get ICI free signal models for $\mathbf{G}_p$ with $p \in \bar{\mathcal{P}}$ by multiplying $\mathbf{z}'^b$ by the pseudo-inverse of the matrix $N^{-\frac{1}{2}}\mathbf{F}_N\hat{\mathbf{V}}'$, where $\hat{\mathbf{V}}'$ contains all the distinct columns of $\mathbf{V}'$.
2) Jointly decode the rest $P - |\bar{\mathcal{P}}|$ SF submatrices by first canceling all of the power from $\mathbf{z}'^b$ in (46), which comes from decoded $\mathbf{G}_p$ for $p \in \bar{\mathcal{P}}$.

Examining the ZF-ML-PIC method, we can see that when $\mathbf{U}'^b$ is singular, only $(P - |\bar{\mathcal{P}}|)$ SF submatrices need to be jointly decoded. So when $|\bar{\mathcal{P}}|$ is larger, the ZF-ML-PIC method has much lower computational complexity than that of jointly decoding $\mathbf{G}_p$ for $1 \le p \le P$. For the ZF-ML-PIC method, we have the following theorem.

Theorem 5: When $\bar{\mathcal{P}}$ is not null, for the permuted SF code $\mathbf{C}'$ described by (29), the ZF-ML-PIC detection method can still achieve the same diversity order as the case without CFOs.

Proof: Assume that without CFOs $\mathbf{C}'$ can achieve diversity order d. According to Theorem 2, we can express the SER/BER of decoding $\mathbf{G}_p$ for $p \in \bar{\mathcal{P}}$ as $P_a = \frac{1}{c_1}\text{SNR}^{-d}$, where c_1 is the coding gain. Then define two Bernoulli random variables I_a and I_b with probability mass functions given by

$$I_a = \begin{cases} 0, & \text{if bits/symbols in } \mathbf{G}_p,\ p \in \bar{\mathcal{P}} \text{ are not all correctly decoded} \\ 1, & \text{otherwise} \end{cases}, \tag{47}$$

and

$$I_b = \begin{cases} 0, & \text{if bits/symbols in } \mathbf{G}_p,\ p \notin \bar{\mathcal{P}}, \text{ are not all correctly decoded} \\ 1, & \text{otherwise} \end{cases}, \tag{48}$$

respectively. Denoting P_b as the BER/SER for decoding $\mathbf{G}_p$ for $p \notin \bar{\mathcal{P}}$, we can express P_b as

$$\begin{aligned} P_b &= \text{P}\{I_b = 0 | I_a = 1\}(1 - \text{P}\{I_a = 0\}) \\ &\quad + \text{P}\{I_b = 0 | I_a = 0\}\text{P}\{I_a = 0\}. \end{aligned} \tag{49}$$

In (49), $\text{P}\{I_b = 0|I_a = 1\}$ is the BER/SER for jointly decoding $\mathbf{G}_p$ for $p \notin \bar{\mathcal{P}}$ in the case that $\mathbf{G}_p$, $p \in \bar{\mathcal{P}}$, are all correctly decoded. From Theorem 3, we know that in this case jointly decoding $\mathbf{G}_p$ for $p \notin \bar{\mathcal{P}}$ can still achieve diversity order d. Thus $\text{P}\{I_b = 0|I_a = 1\}$ can be expressed by $\frac{1}{c_2}\text{SNR}^{-d}$, where c_2 is the coding gain. From the definition of I_a, we also get $\text{P}\{I_a = 0\} = P_a = \frac{1}{c_1}\text{SNR}^{-d}$. Substituting the expressions of $\text{P}\{I_b = 0|I_a = 1\}$ and $\text{P}\{I_a = 0\}$ into (49), we obtain

$$\begin{aligned} P_b &= \frac{1}{c_2}\text{SNR}^{-d} - \frac{1}{c_1 c_2}\text{SNR}^{-2d} \\ &\quad + \text{P}\{I_b = 0|I_a = 0\}\frac{1}{c_1}\text{SNR}^{-d} \\ &< (\frac{1}{c_1} + \frac{1}{c_2})\text{SNR}^{-d}, \end{aligned} \tag{50}$$

where we have utilized the fact $\text{P}\{I_b = 0|I_a = 0\} \le 1$. From (50), we see that the ZF-ML-PIC method can achieve the same diversity order as the case without CFOs for each $\mathbf{G}_p$ for $1 \le p \le P$. ■

For $\mathbf{C}'$ given by (29), both the ZF-ML-Zn and the ZF-ML-PIC methods can achieve the same diversity order as the case without CFOs and their computational complexities are much reduced than jointly considering all the N subcarriers. Compared to the ZF-ML-PIC method, the ZF-ML-Zn approach has a lower computational complexity at the cost of a bandwidth efficiency loss $\frac{M_t n}{N}$.

VI. Simulation Results

In this section, we present some simulation results to verify our analysis on the diversity order achieved by the SF codes $\mathbf{C}$ and $\mathbf{C}'$ with multiple CFOs. Firstly, an $M_t = 2$ system with 8 OFDM subcarriers is simulated. The bandwidth is 20 MHz and BPSK modulation is employed. The channels from relays to the destination node are frequency-selective with two equal power rays $[\tau_m(0), \tau_m(1)] = [0, 0.1]\mu s$ for $1 \le m \le M_t$. We also assume that the destination node has only one receive antenna. For each channel realization, each ε_m is uniformly selected from $[-\varepsilon_{Max}, \varepsilon_{Max}]$. The SF codes proposed in [11] are applied. Coding parameter Γ is set as 2. Thus without CFOs, diversity order 4 can be achieved.

Fig. 4 shows the SER performance of the non-permuted SF codes $\mathbf{C}$ in the presence of multiple CFOs. We can see that when $\varepsilon_{Max} = 0.4$, as diversity order 4 can still be achieved, the same slope of SER curve as that of the case without CFOs is observed. We also note that multiple CFOs cause only a very small coding gain loss, since as ε_{Max} is increased, some nonzero eigenvalues of the difference matrix $\mathbf{U}^b(\mathbf{X}-\hat{\mathbf{X}})\mathbf{\Lambda}(\mathbf{X}-\hat{\mathbf{X}})^{\mathcal{H}}\mathbf{U}^{b\mathcal{H}}$ may approach zero. When $\varepsilon_{Max} = 0.8$, as full diversity cannot always be achieved for each realization of CFOs, the slope is no longer parallel with the curve without CFOs at the high SNR range. Finally, we simulated a special case, i.e., $\varepsilon_1 = 0.6$ and $\varepsilon_2 = -0.4$. From the analysis in Section IV-C, we know that for this special case full diversity cannot be achieved. The simulation result confirms our analysis. In Fig. 4, the slope of its SER curve is obvious less than that of the SER curve without CFOs.

Fig. 5 shows the SER performance of the permuted SF codes $\mathbf{C}'$ in the presence of multiple CFOs. According to Theorem 3, we know that for $P = 2$ and $\Gamma = 2$, the achieved diversity order of $\mathbf{C}'$ is not decreased when $\varepsilon_{Max} < 1.5$. This is confirmed by the simulation results. The SER curve in the case of $\varepsilon_{Max} = 1.4$ has the same slope as that of the case without CFOs. Even in the special case of $\varepsilon_1 = 0.6$ and $\varepsilon_2 = -0.4$, although the ICI matrix $\mathbf{U}'$ is singular, $\mathbf{C}'$ still achieves the same slope of SER curve as that of the case without CFOs. However, when $\varepsilon_{Max} = 1.9$ and $\varepsilon_{Max} = 2.1$, the SER curves are no longer parallel with that of the case without CFOs at the high SNR range, since when $\varepsilon_{Max} > 1.5$ full diversity cannot always be achieved for each realization of CFOs. Finally, we simulated a special case, i.e., $\varepsilon_1 = 1.6$ and $\varepsilon_2 = -1.4$.

By similar analysis as given in Section IV-C, it can be shown that in this special case $\mathbf{C}'$ cannot achieve full diversity. The simulation result confirms our analysis. In Fig.5, the slope of its SER curve is obviously less than that of the SER curve without CFOs.

To investigate the SER performance of the two-stage methods, we consider an $M_t = 2$ system with 64 subcarriers. The channels from relays to the destination node are all frequency-selective fading with two equal power rays and $[\tau_m(0), \tau_m(1)] = [0, 0.5]\mu s$ for $1 \le m \le 2$. The data symbol is QPSK modulated. To achieve full diversity order 4 we set Γ as 2. Fig.6 shows the simulation results. We can see that with multiple CFOs as SNR increases, the OFDM system will quickly suffer from an error floor if we directly decode (referred to as DD) the SF codes by only ignoring all the ICI terms due to CFOs. When $\varepsilon_{Max} = 0.2$, for the ZF-ML method, the performance loss is very small and the same slope of SER curve as that of the case without CFOs is observed. As ε_{Max} is increased from 0.2 to 0.4, the performance of the ZF-ML method is degraded. Since ε_{Max} is still less than 0.5, from Theorem 2, we see that when $\varepsilon_{Max} = 0.4$ the ZF-ML method can still achieve full diversity. This is confirmed by simulation results. Note that ML decoding of all the subcarriers simultaneously is the optimal method to achieving full diversity. However, when N is 64 and QPSK modulation is employed, it leads to a great burden to the system even for efficient ML detection methods such as the sphere decoding method.

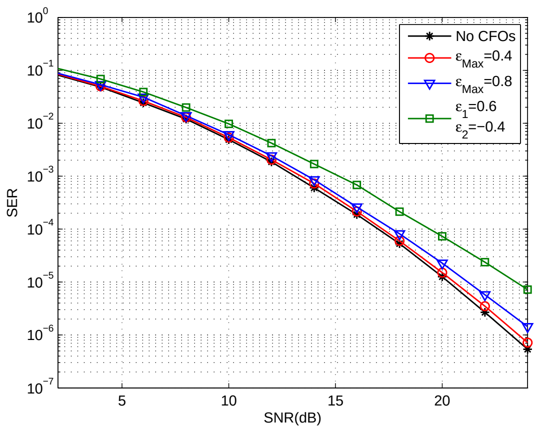

Fig. 4. Performance of the SF codes $\mathbf{C}$ in the presence of multiple CFOs

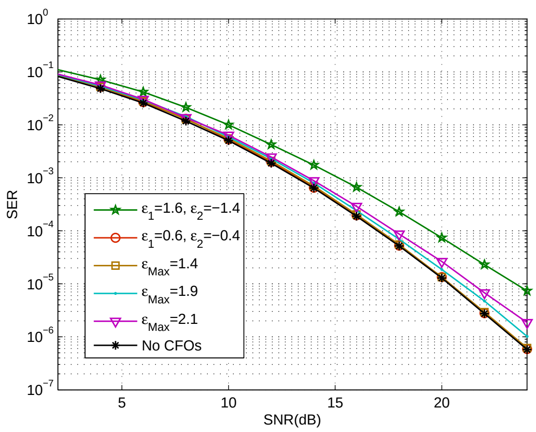

Fig. 5. Performance of the permuted SF codes $\mathbf{C}'$ in the presence of multiple CFOs.

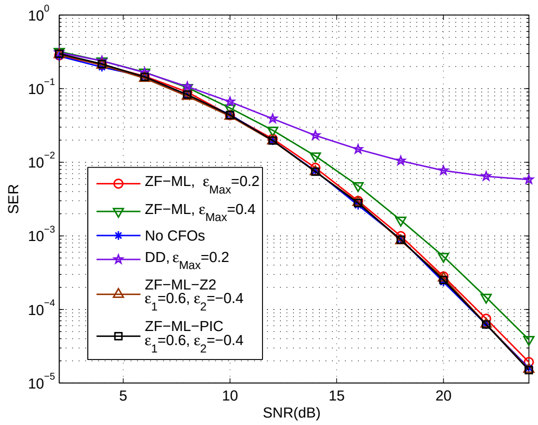

Fig. 6. Performance of the two-stage detection methods.

Finally, we consider the permuted SF code $\mathbf{C}'$ in the case of $\varepsilon_1 = 0.6$ and $\varepsilon_2 = -0.4$, which is possible when $\varepsilon_{Max} \ge 0.5$. As illustrated, the non-permuted SF code $\mathbf{C}$ in this case cannot achieve full diversity any more. Since the ICI matrix $\mathbf{U}'$ is singular, the ZF-ML method cannot be directly applied. To still achieve full diversity, we use the ZF-ML-Z2 and ZF-ML-PIC methods. For the ZF-ML-Z2 method, the 30th, 31st, 62nd and 63rd subcarriers are not used. According to Theorem 4, full diversity can be achieved as long as both $|\varepsilon_1|$ and $|\varepsilon_2|$ are smaller than 1.5. For the ZF-ML-PIC method, $\mathbf{G}_1$ and $\mathbf{G}_{16}$ need to be jointly decoded. The simulation results are consistent with our previous analysis. In Fig. 6, it is apparent that both of these two detection methods achieve the same diversity order as that without CFOs.

VII. Conclusion

In this paper, we investigate the effect of multiple CFOs in a cooperative OFDM based system on a family of SF codes proposed in [11]. By treating the CFOs as a part of the SF code matrix, we showed that if ε_{Max} is less than 0.5, the full diversity order for the SF codes are not affected by the multiple CFOs in the SF coded OFDM cooperative system. We further prove that this full diversity property can still be preserved if the zero forcing (ZF) method is used to equalize the multiple CFOs. In order to improve the robustness of the SF codes to multiple CFOs, we proposed a permutation method to enable the SF codes to achieve full diversity even when $\varepsilon_{max} \ge 0.5$. Furthermore, two full diversity achievable detection methods, namely the ZF-ML-Zn and ZF-ML-PIC, have been introduced, both of which are applicable to cases when ICI matrix is singular. All these imply that the SF codes proposed in [11] for MIMO-OFDM systems are robust to both timing errors and CFOs in a cooperative system.

Appendix A
Proofs of Proposition 1 and Properties of $\mathbf{V}'$

A. *Proof of Proposition 1*

From (35), we know that $\mathbf{V}'$ is singular, if and only if there exist at least three integers t, i and l which satisfy

$$\varepsilon_{m_l} - \varepsilon_{m_i} = tN + i - l, \tag{I1}$$

for $0 \le i < l \le N-1$. As $0 \le i < l \le N-1$, it follows that $-N+1 \le i-l \le -1$. Consequently, we have

$$tN - N + 1 \le tN + i - l \le tN - 1. \tag{I2}$$

When (I1) holds, substituting (I1) into (I2), we obtain

$$tN - N + 1 \le \varepsilon_{m_l} - \varepsilon_{m_i} \le tN - 1. \tag{I3}$$

On the other hand, provided $|\varepsilon_m| < \frac{(P-1)\Gamma+1}{2}$ and $P > 1$, we have

$$-N < -(P-1)\Gamma - 1 < \varepsilon_{m_l} - \varepsilon_{m_i} < (P-1)\Gamma + 1 < N. \tag{I4}$$

Then let us examine (I1) in great detail when it holds.

Firstly, if $\varepsilon_{m_l} - \varepsilon_{m_i} = 0$, from (I1) we have $tN + i - l = 0$ and then $t = \frac{l-i}{N}$. On the other hand, since $-N+1 \le i-l \le -1$, we can get $\frac{1}{N} \le \frac{l-i}{N} \le 1 - \frac{1}{N}$. Therefore, we obtain $\frac{1}{N} \le t \le 1 - \frac{1}{N}$, which is contradictory to the fact that t is an integer. Thus $\varepsilon_{m_l} - \varepsilon_{m_i}$ cannot be equal to zero.

Secondly, if $\varepsilon_{m_l} - \varepsilon_{m_i} < 0$, given (I4), $\varepsilon_{m_l} - \varepsilon_{m_i}$ must be in the set $\{-(P-1)\Gamma, \cdots, -1\}$, since when (I1) holds, $\varepsilon_{m_l} - \varepsilon_{m_i}$ should be an integer. To let both (I3) and (I4) hold, it is required that $tN - N + 1 < 0$ and $tN - 1 > -N$, which lead to $t < 1 - \frac{1}{N}$ and $t > -1 + \frac{1}{N}$, respectively. Considering the fact that t is an integer, we can derive that t can only be equal to 0. Thus we have $\varepsilon_{m_l} - \varepsilon_{m_i} = i - l$. From (34), we know that all the columns of $\mathbf{V}'_m$ are related to ε_m. In conjunction with $i < l$ and $\varepsilon_{m_i} \ne \varepsilon_{m_l}$, we can obtain $1 \le m_i < m_l \le M_t$ and then $m_l - m_i \ge 1$. On the other hand, since $\varepsilon_{m_l} - \varepsilon_{m_i}$ is in the set $\{-(P-1)\Gamma, \cdots, -1\}$ and $\varepsilon_{m_l} - \varepsilon_{m_i} = i - l$, we can get

$$1 \le l - i \le (P-1)\Gamma. \tag{I5}$$

From (34), we have

$$\begin{aligned} m_l - m_i &= \left\lceil \frac{l+1}{P\Gamma} \right\rceil - \left\lceil \frac{i+1}{P\Gamma} \right\rceil \\ &\le \left\lceil \frac{i+(P-1)\Gamma+1}{P\Gamma} \right\rceil - \left\lceil \frac{i+1}{P\Gamma} \right\rceil \\ &\le \left\lceil \frac{i+1}{P\Gamma} \right\rceil + 1 - \left\lceil \frac{i+1}{P\Gamma} \right\rceil \\ &= 1, \end{aligned} \tag{I6}$$

where the second inequality follows from the fact that $0 < \frac{(P-1)\Gamma}{P\Gamma} < 1$. Therefore, $m_l - m_i$ can only be equal to 1.

Thirdly, if $\varepsilon_{m_l} - \varepsilon_{m_i} > 0$, by the similar analysis to that for $\varepsilon_{m_l} - \varepsilon_{m_i} < 0$, it is not hard to obtain that when (I1) holds, $\varepsilon_{m_l} - \varepsilon_{m_i} \in \{1, 2, \cdots, (P-1)\Gamma\}$, $1 \le m_i < m_l \le M_t$ and t can only be equal to 1. From $t = 1$, we have $\varepsilon_{m_l} - \varepsilon_{m_i} = N + i - l$. Since $\varepsilon_{m_l} - \varepsilon_{m_i}$ is in the set $\{1, 2, \cdots, (P-1)\Gamma\}$, we get

$$1 \le N + i - l \le (P-1)\Gamma. \tag{I7}$$

From $1 \le m_i \le M_t - 1$ and (34), we also have

$$0 \le i \le (M_t - 1)P\Gamma - 1, \tag{I8}$$

since $\mathbf{v}'_i$ is one column of $\mathbf{V}'_m$ for $1 \le m \le M_t - 1$. From (I7), (I8) and $l \le N-1$, it is not hard to get the inequality

$$N - (P-1)\Gamma \le l \le N - 1. \tag{I9}$$

Then according to (34), we have $m_l = M_t$. Adding both sides of (I9) to that of (I7) and using $i \ge 0$, we can obtain

$$0 \le i \le (P-1)\Gamma - 1, \tag{I10}$$

which leads to $m_i = 1$.

By summarizing results of the above analysis, we can get Proposition 1. ■

B. *Proofs of Properties of* $\mathbf{V}'$

1) Proof of Property 1: Firstly, for the case 1 of Proposition 1, we need to calculate the lower bound of i. Based on the fact that $\mathbf{v}'_i$ is one column of $\mathbf{V}'_{m_i}$, we can get that $(m_i - 1)P\Gamma \le i \le m_i P\Gamma - 1$. As $m_l - m_i = 1$, we then have

$$(m_l - 2)P\Gamma \le i \le (m_l - 1)P\Gamma - 1. \tag{I11}$$

Substituting (I11) into (I5), we obtain

$$(m_l - 2)P\Gamma + 1 \le l \le m_l P\Gamma - 1 - \Gamma. \tag{I12}$$

As $\mathbf{v}'_l$ is one column of $\mathbf{V}'_{m_l}$, we immediately have $(m_l - 1)P\Gamma \le l$. Thus from (I12) we finally obtain

$$(m_l - 1)P\Gamma \le l \le m_l P\Gamma - 1 - \Gamma. \tag{I13}$$

Then from (I5), (I11), (I13) and $m_i = m_l - 1$, we can easily get the bounds of i as

$$m_i P\Gamma - (P-1)\Gamma \le i \le m_i P\Gamma - 1. \tag{I14}$$

Secondly, for the case 2 of Proposition 1, we need calculate the lower bond of l, where $m_l = M_t$. It has been shown in (I9) as $l \ge N - (P-1)\Gamma = M_t P\Gamma - (P-1)\Gamma$.

2) Proof of Property 2: Assume $f_m(t_{m,1}) > (m)_{M_t} P\Gamma$. Then it is easy to show that $t_{m,1} - 1$ and $f_m(t_{m,1}) - 1$ are also a pair of integers satisfying Proposition 1. So we have $\mathbf{v}'_{t_{m,1}-1} = \mathbf{v}'_{f_m(t_{m,1})-1}$ and thus $(t_{m,1} - 1) \in \mathcal{B}_m$. As this is contradictory to the assumption that $t_{m,1}$ is the minimum element of $\mathcal{B}_m$, we have $f_m(t_{m,1}) \le (m)_{M_t} P\Gamma$. On the other hand, since $f_m(t_{m,1})$ cannot be smaller than $(m)_{M_t} P\Gamma$, we obtain our conclusion.

3) Proof of Property 3: From the fact that $t_{m,1}$ and $f_m(t_{m,1})$ are a pair of integers satisfying Proposition 1, it is not hard to get that if $t_{m,1} < mP\Gamma - 1$, $t_{m,1} + k$ and $f_m(t_{m,1}) + k$ for $1 \le k \le mP\Gamma - 1 - t_{m,1}$ are also a pair of integers that satisfies Proposition 1. Therefore, $\mathcal{B}_m$ has the form $\mathcal{B}_m = \{t_{m,1}, t_{m,1} + 1, \cdots, mP\Gamma - 1\}$.

4) Proof of Property 4: Directly comes from the upper bound of l in (I12) and that of i in (I10).

5) Proof of Property 5: Directly follows from the *Property* 2 and *Property* 3 of $\mathbf{V}'$ and the relationship between $\mathcal{B}_m$ and $\bar{\mathcal{B}}_m$.

6) Proof of Property 6: In this proof, we consider three different cases of m, i.e., $m = 1$, $2 \le m \le M_t - 1$ and $m = M_t$. Here only the derivation for the case of $2 \le m \le M_t - 1$ is given. For the other two cases, the same conclusion can be obtained by following the similar analysis strategy.

As $\bar{\mathcal{B}}_m$ is not null, from *Property* 5 of $\mathbf{V}'$, we know that $\bar{t}_{m,\bar{T}_m} = (m-1)P\Gamma + \bar{T}_m - 1 = f_{m-1}(t_{m-1,T_{m-1}})$. From *Property* 3 of $\mathbf{V}'$, we have $t_{m-1,T_{m-1}} = (m-1)P\Gamma - 1$. Thus, we get $\bar{t}_{m,\bar{T}_m} = f_{m-1}((m-1)P\Gamma - 1)$. Then by using (42), $\bar{t}_{m,\bar{T}_m}$ can be expressed by

$$\bar{t}_{m,\bar{T}_m} = (m-1)P\Gamma - 1 - \varepsilon_m + \varepsilon_{m-1}. \tag{I15}$$

On the other hand, based on *Property* 2 of $\mathbf{V}'$, we have $f_m(t_{m,1}) = mP\Gamma$. From (42), we get

$$mP\Gamma = t_{m,1} - \varepsilon_{m+1} + \varepsilon_m. \tag{I16}$$

Adding each side of (I16) to that of (I15), we get

$$\varepsilon_{m+1} - \varepsilon_{m-1} = t_{m,1} - \bar{t}_{m,\bar{T}_m} - P\Gamma - 1. \tag{I17}$$

Given $|\varepsilon_m| < \frac{(P-1)\Gamma+1}{2}$, we have $\varepsilon_{m+1} - \varepsilon_{m-1} > -(P-1)\Gamma - 1$. Substituting this inequality into (I17), we can get $t_{m,1} - \bar{t}_{m,\bar{T}_m} > \Gamma$. As $t_{m,1} - \bar{t}_{m,\bar{T}_m}$ is an integer, we finally obtain $t_{m,1} - \bar{t}_{m,\bar{T}_m} \ge \Gamma + 1$.

7) Proof of Property 7: Assume that $\mathcal{B}_m \ne \phi$ for $1 \le m \le M_t$. From the case 1 of Proposition 1, we have $-(P-1)\Gamma \le \varepsilon_m - \varepsilon_{m-1} \le -1$ for $2 \le m \le M_t$. Then we can obtain $\sum_{m=2}^{M_t} -(P-1)\Gamma \le \sum_{m=2}^{M_t} \varepsilon_m - \varepsilon_{m-1} \le \sum_{m=2}^{M_t} -1$, which leads to

$$-(M_t - 1)(P-1)\Gamma \le \varepsilon_{M_t} - \varepsilon_1 \le -(M_t - 1). \tag{I18}$$

On the other hand, since $\mathcal{B}_{M_t} \ne \phi$, from the case 2 of Proposition 1, we get that $1 \le \varepsilon_{M_t} - \varepsilon_1 \le (P-1)\Gamma$, which is contradictory to (I18). Therefore, there must exist $m^\dagger$ so that $\mathcal{B}_{m^\dagger} = \phi$.

8) Proof of Property 8: From *Property* 7, we know that there must exist $n^\dagger$ with $1 \le n^\dagger \le M_t$ so that $\mathcal{B}_{n^\dagger} = \phi$. From the relationship between $\mathcal{B}_n$ and $\bar{\mathcal{B}}_{(n)_{M_t}+1}$ (*Property* 5 of $\mathbf{V}'$), we can deduce that $\mathcal{B}_n = \bar{\mathcal{B}}_{(n)_{M_t}+1} = \phi$. Therefore, *Property* 8 holds. ■

APPENDIX B
PROOF OF LEMMA 1

Above all, we remind that all the discussions in this Appendix are under the condition $|\varepsilon_m| < \frac{(P-1)\Gamma+1}{2}$ for $1 \le m \le M_t$ and $P > 1$. Hence Proposition 1 and the properties of $\mathbf{V}'$ described in Section V-D hold. To simplify notation, we use $\mathbf{r}$ to stand for $\sum_{m=1}^{n_1} \mathbf{V}'_m \Delta\bar{\mathbf{Y}}_m \mathbf{a}_m + \sum_{m=n_2}^{M_t} \mathbf{V}'_m \Delta\bar{\mathbf{Y}}_m \mathbf{a}_m$.

As $\mathbf{V}'_{n_2}$ is the $(n_2 - 1)$th $N \times P\Gamma$ submatrix of $\mathbf{V}'$, it contains the columns of $\mathbf{V}'$ from $\mathbf{v}_{(n_2-1)P\Gamma}$ to $\mathbf{v}_{n_2 P\Gamma - 1}$, where $\mathbf{v}'_l$ stands for lth column of $\mathbf{V}'$ for $0 \le l \le N-1$. According to Proposition 1, we know that when $\mathbf{V}'$ is singular, some columns of $\mathbf{V}'_{n_2}$ may be repeated in $\mathbf{V}'_{(n_2-2)_{M_t}+1}$ or $\mathbf{V}'_{(n_2)_{M_t}+1}$, where $(\ )_n$ means the modular operation. Further from *Property* 6 of $\mathbf{V}'$, we can see that the columns of $\mathbf{V}'_{n_2}$ can only be repeated in $\mathbf{V}'_{(n_2-2)_{M_t}+1}$ or $\mathbf{V}'_{(n_2)_{M_t}+1}$.

From the definition of $\bar{B}_{n_2}$ in (41), we know that when $\bar{B}_{n_2} = \phi$, no columns of $\mathbf{V}'_{n_2}$ are repeated in $\mathbf{V}'_{(n_2-2)_{M_t}+1}$. On the other hand, according to *Property* 1 of $\mathbf{V}'$, when $B_{n_2} \ne \phi$, we can obtain $t_{n_2,1} - (n_2 - 1)P\Gamma \ge \Gamma$, which means that the first Γ column of $\mathbf{V}'_{n_2}$ cannot be repeated in $\mathbf{V}'_{(n_2)_{M_t}+1}$, since $\mathbf{v}'_{(n_2-1)P\Gamma}$ is the first column of $\mathbf{V}'_{n_2}$. Therefore, when $\bar{B}_{n_2} = \phi$, at least the first Γ columns of $\mathbf{V}'_{n_2}$ are unique in $\mathbf{V}'_m$ for $1 \le m \le n_1$ and $n_2 \le m \le M_t$.

Then let us turn to the condition $n_2 \ge n_1 + 2$. From $n_2 \ge n_1 + 2$ and $0 \le n_1 < n_2 \le M_t$, it is easy to obtain $n_2 > (n_2 - 2)_{M_t} + 1 \ge n_1 + 1$. Hence, $\mathbf{V}'_{(n_2-2)_{M_t}+1}$ does not exist in the expression of $\mathbf{r}$. As the first Γ columns of $\mathbf{V}'_{n_2}$ can only be repeated in $\mathbf{V}'_{(n_2-2)_{M_t}+1}$, in the case of $n_2 \ge n_1 + 2$ we can obtain the same conclusion as that in the case of $\bar{B}_{n_2} = \phi$, i.e., at least the first Γ columns of $\mathbf{V}'_{n_2}$ are unique in $\mathbf{V}'_m$ for $1 \le m \le n_1$ and $n_2 \le m \le M_t$.

By using the block matrix multiplication rule, we rewrite $\mathbf{r}$ as

$$\mathbf{r} = \sum_{m=1}^{n_1} \sum_{p=1}^{P} \mathbf{V'}_m^p \Delta\bar{\mathbf{Y}}_m^p \mathbf{a}_m + \sum_{m=n_2}^{M_t} \sum_{p=1}^{P} \mathbf{V'}_m^p \Delta\bar{\mathbf{Y}}_m^p \mathbf{a}_m, \tag{II1}$$

where $\mathbf{V'}_m^p$ denotes the $(p-1)$th $N \times \Gamma$ submatrix of $\mathbf{V}'_m$ and $\bar{\mathbf{Y}}_m^p$ denotes the $(p-1)$th $\Gamma \times \min(\Gamma, L_m)$ submatrix of $\bar{\mathbf{Y}}_m$. Let $p_{n_2}^\dagger$ with $1 \le p_{n_2}^\dagger \le P$ be the maximum number so that the columns of $\mathbf{V'}_{n_2}^p$, $1 \le p \le p_{n_2}^\dagger$, are unique in $\mathbf{V}'_m$ for $1 \le m \le n_1$ and $n_2 \le m \le M_t$. As we have discussed, when either $\bar{B}_{n_2} = \phi$ or $n_2 \ge n_1 + 2$, the first Γ columns of $\mathbf{V}'_{n_2}$ are unique in $\mathbf{V}'_m$ for $1 \le m \le n_1$ and $n_2 \le m \le M_t$. Therefore, $p_{n_2}^\dagger$ is at least equal to 1. Then we rewrite (II1) as (II2) at the top of next page and regard $\mathbf{r}$ as a linear combination of the columns of $\mathbf{V}'_m$ with the coefficient vectors $\Delta\bar{\mathbf{Y}}_m \mathbf{a}_m$ for $1 \le m \le n_1$ and $n_2 \le m \le M_t$.

Let us first look at the second term in the right-hand side of (II2) which is related to $\mathbf{V'}_{n_2}^p$ for $1 \le p \le p_{n_2}^\dagger$. As $\mathbf{V}'$ is a Vandermonde matrix, all the distinct columns of $\mathbf{V}'_m$ for $1 \le m \le n_1$ and $n_2 \le m \le M_t$ are linearly independent. Further considering the fact that the columns of $\mathbf{V'}_{n_2}^p$ for $1 \le p \le p_{n_2}^\dagger$ are unique in the columns of $\mathbf{V}'_m$ for $1 \le m \le n_1$ and $n_2 \le m \le M_t$, to let $\mathbf{r} = \mathbf{0}_N$, we must have

$$\Delta\bar{\mathbf{Y}}_{n_2}^p \mathbf{a}_{n_2} = \mathbf{0}_\Gamma \quad \text{for } 1 \le p \le p_{n_2}^\dagger. \tag{II4}$$

Assume that there exists p_0 with $1 \le p_0 \le p_{n_2}^\dagger$ so that $\mathbf{G}_{p_0} \ne \hat{\mathbf{G}}_{p_0}$. Then from (II4), we get $\Delta\bar{\mathbf{Y}}_{n_2}^{p_0} \mathbf{a}_{n_2} = \mathbf{0}_\Gamma$. From Proposition 2, we know that $\Delta\bar{\mathbf{Y}}_{n_2}^{p_0}$ has full column rank $\min(\Gamma, L_{n_2})$. As a consequence, we can obtain that $\Delta\bar{\mathbf{Y}}_{n_2}^{p_0} \mathbf{a}_{n_2} = \mathbf{0}_\Gamma$, only if $\mathbf{a}_{n_2} = \mathbf{0}_{\min(\Gamma, L_{n_2})}$. Thus if there exists p_0 with $1 \le p_0 \le p_{n_2}^\dagger$ so that $\mathbf{G}_{p_0} \ne \hat{\mathbf{G}}_{p_0}$, then $\mathbf{r} = \mathbf{0}_N$, only if $\mathbf{a}_{n_2} = \mathbf{0}_{\min(\Gamma, L_{n_2})}$. Note when $\mathcal{B}_{n_2} = \phi$, we have $p_{n_2}^\dagger = P$, which means that Lemma 1 holds for the case of $\mathcal{B}_{n_2} = \phi$.

When $\mathcal{B}_{n_2} \ne \phi$, we have $1 \le p_{n_2}^\dagger < P$. Following the above discussions, we only need to consider the case that $\mathbf{G}_p = \hat{\mathbf{G}}_p$ for $1 \le p \le p_{n_2}^\dagger$, which leads to $\Delta\bar{\mathbf{Y}}_m^p =$

$$\mathbf{r} = \sum_{m=1}^{n_1}\sum_{p=1}^{P}\mathbf{V'}_m^p\Delta\bar{\mathbf{Y}}_m^p\mathbf{a}_m + \sum_{p=1}^{p_{n_2}^{\dagger}}\mathbf{V'}_{n_2}^p\Delta\bar{\mathbf{Y}}_{n_2}^p\mathbf{a}_{n_2} + \sum_{p=p_{n_2}^{\dagger}+1}^{P}\mathbf{V'}_{n_2}^p\Delta\bar{\mathbf{Y}}_{n_2}^p\mathbf{a}_{n_2} + \sum_{p=1}^{P}\mathbf{V'}_{n_2+1}^p\Delta\bar{\mathbf{Y}}_{n_2+1}^p\mathbf{a}_{n_2+1} + \sum_{m=3}^{M_t}\sum_{p=1}^{P}\mathbf{V'}_m^p\Delta\bar{\mathbf{Y}}_m^p\mathbf{a}_m \quad \text{(II2)}$$

$$\mathbf{r} = \sum_{m=1}^{n_1}\sum_{p=p_{n_2}^{\dagger}+1}^{P}\mathbf{V}'_m\Delta\bar{\mathbf{Y}}_m^p\mathbf{a}_m + \mathbf{V'}_{n_2}^{p_{n_2}^{\dagger}+1}\Delta\bar{\mathbf{Y}}_{n_2}^{p_{n_2}^{\dagger}+1}\mathbf{a}_{n_2} + \sum_{p=p_{n_2}^{\dagger}+2}^{P}\mathbf{V'}_{n_2}^p\Delta\bar{\mathbf{Y}}_{n_2}^p\mathbf{a}_{n_2} + \sum_{p=p_{n_2}^{\dagger}+1}^{P}\mathbf{V'}_{n_2+1}^p\Delta\bar{\mathbf{Y}}_{n_2+1}^p\mathbf{a}_{n_2+1} + \sum_{m=3}^{M_t}\sum_{p=p_1^{\dagger}+1}^{P}\mathbf{V'}_m^p\Delta\bar{\mathbf{Y}}_m^p\mathbf{a}_m. \quad \text{(II3)}$$

$\mathbf{0}_{\Gamma\times\min(\Gamma,L_m)}$ for $1 \le p \le p_{n_2}^{\dagger}$ and $1 \le m \le M_t$. Substituting this fact into (II2), we have (II3) at the top of next page. From the definitions of $p_{n_2}^{\dagger}$ and $\mathcal{B}_{n_2}$ in (40), we have $t_{n_2,1} \le (n_2-1)P\Gamma+(p_{n_2}^{\dagger}+1)\Gamma-1$, since $\mathbf{v}'_{t_{n_2,1}}$ should be one column of $\mathbf{V}_{n_2}^{p_{n_2}^{\dagger}+1}$. Then from *Property* 3 and *Property* 5 of $\mathbf{V}'$, the last $(n_2-1)P\Gamma+(p_{n_2}^{\dagger}+1)\Gamma-t_{n_2,1}$ columns of $\mathbf{V'}_{n_2}^{p_{n_2}^{\dagger}+1}$ are equal to the first $(n_2-1)P\Gamma+(p_{n_2}^{\dagger}+1)\Gamma-t_{n_2,1}$ columns of $\mathbf{V'}_{n_2+1}^{1}$. However, due to $\Delta\bar{\mathbf{Y}}_{n_2+1}^1 = \mathbf{0}_{\Gamma\times\min(\Gamma,L_{n_2+1})}$, the term $\mathbf{V'}_{n_2+1}^{1}\Delta\bar{\mathbf{Y}}_{n_2+1}^1\mathbf{a}_{n_2+1}$ does not exist in the right-hand side of (II3). Note that from *Property* 6 of $\mathbf{V}'$, we know in $\mathbf{V}'$ there are no more than 2 equal columns. It follows that in (II3) only the term $\mathbf{V'}_{n_2}^{p_{n_2}^{\dagger}+1}\Delta\bar{\mathbf{Y}}_{n_2}^{p_{n_2}^{\dagger}+1}\mathbf{a}_{n_2}$ is related to the columns of $\mathbf{V'}_{n_2}^{p_{n_2}^{\dagger}+1}$. Since all the distinct columns of $\mathbf{V}'$ are linearly independent, to let $\mathbf{r} = \mathbf{0}_N$ in (II3), we must have $\Delta\bar{\mathbf{Y}}_{n_2}^{p_{n_2}^{\dagger}+1}\mathbf{a}_{n_2} = \mathbf{0}_\Gamma$. If $\mathbf{G}_{p_{n_2}^{\dagger}+1} \ne \hat{\mathbf{G}}_{p_{n_2}^{\dagger}+1}$, due to Proposition 2, $\Delta\bar{\mathbf{Y}}_{n_2}^{p_{n_2}^{\dagger}+1}$ has full column rank $\min(\Gamma, L_{n_2})$. As a consequence, if $\mathbf{G}_{p_{n_2}^{\dagger}+1} \ne \hat{\mathbf{G}}_{p_{n_2}^{\dagger}+1}$, then $\Delta\mathbf{Y}_{n_2}^{p_{n_2}^{\dagger}+1}\mathbf{a}_{n_2} = \mathbf{0}_\Gamma$, only if $\mathbf{a}_{n_2} = \mathbf{0}_{\min(\Gamma,L_{n_2})}$. Combined with previous analysis results, we can see that if there exists p_0 for $1 \le p_0 \le p_{n_2}^{\dagger} + 1$ so that $\mathbf{G}_{p_0} \ne \hat{\mathbf{G}}_{p_0}$, then $\mathbf{r} = \mathbf{0}_N$, only if $\mathbf{a}_{n_2} = \mathbf{0}_{\min(\Gamma,L_{n_2})}$.

Then to show that given $\mathbf{X}' \ne \hat{\mathbf{X}}'$ in (31), $\mathbf{r} = \mathbf{0}_N$, only if $\mathbf{a}_{n_2} = \mathbf{0}_{\min(\Gamma,L_{n_2})}$, we need to consider the case of $\mathbf{G}_p = \hat{\mathbf{G}}_p$ for $1 \le p \le p_{n_2}^{\dagger} + 1$. By substituting $\Delta\bar{\mathbf{Y}}_m^p = \mathbf{0}_{\Gamma\times\min(\Gamma,L_m)}$ for $1 \le p \le p_{n_2}^{\dagger} + 1$, $1 \le m \le n_1$ and $n_2 \le m \le M_t$ into (II2), we get an analogous condition to what we do in the previous case of $\mathbf{G}_p = \hat{\mathbf{G}}_p$ for $1 \le p \le p_{n_2}^{\dagger}$. Due to the fact that $\Delta\bar{\mathbf{Y}}_{n_2+1}^p = \mathbf{0}_{\Gamma\times\min(\Gamma,L_{n_2+1})}$ for $1 \le p \le p_{n_2}^{\dagger} + 1$, only the term $\mathbf{V'}_{n_2}^{p_{n_2}^{\dagger}+2}\Delta\bar{\mathbf{Y}}_{n_2}^{p_{n_2}^{\dagger}+2}\mathbf{a}_{n_2}$ is related to the columns of $\mathbf{V'}_{n_2}^{p_{n_2}^{\dagger}+2}$ in $\mathbf{r}$. Consequently, we can get that once there exists p_0 with $1 \le p_0 \le p_{n_2}^{\dagger} + 2$ so that $\mathbf{G}_{p_0} \ne \hat{\mathbf{G}}_{p_0}$, then $\mathbf{r} = \mathbf{0}_N$, only if $\mathbf{a}_{n_2} = \mathbf{0}_{\min(\Gamma,L_{n_2})}$. By repeating this analysis step for additional $P - p_{n_2}^{\dagger} - 2$ times, we can finally get as expected. ■

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